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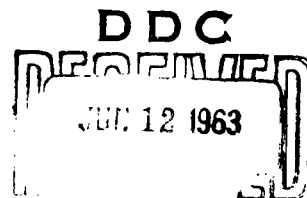
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ON SOME PROBLEMS IN THE THEORY OF PARTICLE
COUNTING AND THE INFINITELY MANY SERVER QUEUE

by

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Introduction

In recent years renewal theory has become one of the most powerful tools of the applied probabilist. In particular, it plays a prominent role in the analysis of the behavior of type I and type II particle counters.

There is an integral formula (due to R. Pyke) for the distribution of the time between successive registrations with a Type I counter under the assumptions that the particles arrive according to a general recurrent process and that the counter has a random dead time. Unfortunately there is no such simple formula in the case of the Type II counter. In chapter I, I give a resume of the renewal theoretic approach to particle counting problems and a shorter proof of Pyke's formula.

The physical literature has been mainly concerned with the case of Poisson-arrivals. The need for considering general recurrent input is genuine, however, because scaling circuits and other devices used by physicists destroy the Poissonian nature of the input (arrival) process. For instance, if we count every r -th arrival in a Poisson process we actually have an Erlang- r input process (i.e., the time between successive arrivals has an Erlang- r distribution).

In his investigations of telephone traffic and Type II particle counters Professor L. Takács discussed both the distribution of $\nu(t) =$

the number of registered particles at time t and of $\eta(t)$ = the number of impulses present in the machine at time t . He formulated a recurrence system of integral equations for the binomial moments of the distribution of $\eta(t)$ and solved them in the case where the dead time (impulse time) produced by each particle is exponentially distributed and the input is recurrent. Although the random variable $\nu(t)$ is of primary concern in particle counting, $\eta(t)$ is the essential random variable in the theory of infinitely many server queueing systems as $\eta(t)$ is precisely the size of the queue at time t .

At the suggestion of Professor L. Takács, I attempted to utilize his methods to solve the more realistic problem where the dead time is allowed to have Erlang- r distribution $r > 1$, rather than the exponential. Unfortunately, the equations become quite involved and I succeeded in determining only the Laplace transforms of the number of impulses present in the counter at time t for Erlang-2 dead times. Luckily, I discovered another class of distributions which are more tractable with respect to these problems, even though they are not easily handled. The distributions I use are "max- m " distributions, the maximum of m exponential distributions each with parameter μ .

After determining the number of impulses present in the type II counter (the queue size problem) I turn to the main problem of particle counting, the behavior of $\nu(t)$. Here is where the "max- m " distributions make the problem manageable. In order to obtain the mean time between consecutive registrations consider the following bulk-queueing problem: Suppose that the particles arrive in batches of size m , each particle independently produces an impulse whose length is exponentially distributed with parameter μ . Then if we allow the 2 stochastic processes:

- 1) Recurrent input; max-m impulse time and
- 2) Recurrent input; batch arrivals of size m with each particle producing an exponentially distributed impulse to occur simultaneously we see that the counter is free at the same time in both processes. Thus if we let $\delta(t)$ = the number of impulses present at time t in the second process, in general $\delta(t) \neq \gamma(t)$ but at any time t when $\gamma(t) = 0$ $\delta(t)$ also will equal 0 and conversely.

In chapter 4, I determine the binomial moments of the ergodic distribution of the imbedded Markov Chain for the batch arrival problem. Using the limiting distribution of this chain and Wald's Fundamental Identity of Sequential Analysis we can determine the mean time \hat{m} between successive registrations with max-m dead time. By the elementary renewal theorem the number of "counts" at time t is asymptotically equal to $\frac{t}{\hat{m}}$.

Chapter 5 is concerned with two attempts to obtain formulas for the variance of the time between consecutive registrations. I was unable to obtain a formula in the case of general inter-arrival times, but I give an approximate result for Erlang-r inter-arrival distributions if r is large.

My final chapter is devoted to a modified version of a problem discussed by Professor W.M. Hirsch concerning the application of queueing theory to missile defense systems. I treat the problem of attacking a well fortified base (one with infinitely many missile batteries). Because of the simplicity of the model assumed we were able to introduce and calculate explicitly a loss function which yields a reasonable criteria for evaluating the effectiveness of certain attack and defense strategies. It is hoped that this type of loss function will be "calculable" in more complex situations.

Acknowledgements:

I wish to express my sincere gratitude to my advisor Professor L. Takács for suggesting this field of research and for valuable advice and encouragement while these investigations were in progress. I also wish to thank Professor C.L. Mallows for many stimulating discussions on these and other problems.

Chapter I

Basic Renewal theory and the Type I Counter

The well known Poisson process is a model for an integer valued random process $\{\xi(t) ; t \geq 0\}$ which counts the number of random events occurring in the time interval $(0, t]$. Usually the events are represented by the times τ_1, τ_2, \dots of their occurrence. The random variables $\theta_1 = \tau_1, \theta_2 = \tau_2 - \tau_1, \dots, \theta_n = \tau_n - \tau_{n-1}, \dots$ are called the successive inter-arrival times. More generally if the time differences are assumed to be identically distributed positive random variables with a common distribution function $F(x)$, and if we denote by ξ_t the number of events which occur in the time interval $(0, t]$, then we say that $\{\theta_n, \xi_t\}$ forms a recurrent (or renewal) process.

Notation: 1) $P[\theta_n \leq x] = F(x)$

$$2) E(\theta) = \mu = \int_0^{\infty} x dF(x)$$

$$3) V(\theta) = \sigma^2 = \int_0^{\infty} (x - \mu)^2 dF(x)$$

$$4) \phi(s) = \int_0^{\infty} e^{-sx} dF(x)$$

Definition: The renewal function (or mean value function) of the renewal process is defined by

$$m(t) = E[\xi(t)] = \sum_{n=0}^{\infty} nP[\xi_t = n] .$$

Thus $m(t)$ is the expected number of events occurring in the interval $(0, t]$.

The role of the renewal function, both in theory and practice, can hardly be overstressed. According to W.L. Smith [Ref. 1 page 246]

"... in most applications of renewal theory a knowledge of the renewal function $m(t)$ or even a knowledge of its asymptotic behavior for large values of t , answers most of the questions we are likely to ask."

In fact knowing $m(t)$ we can determine $F(x)$ from

$$5) \quad \int_0^{\infty} e^{-st} dm(t) = \frac{\phi(s)}{1-\phi(s)}$$

The most basic results of renewal theory will now be stated.

Theorem 1 (Elementary Renewal Theorem)

$$6) \quad \lim_{t \rightarrow \infty} \frac{m(t)}{t} = \frac{1}{\lambda} \quad \text{where } \lambda = E(\theta) \leq \infty$$

and the limit $1/\lambda$ is interpreted as 0 if $\lambda = \infty$.

Theorem 2: If $d^2(t)$ denotes the variance of ξ_t and if $\sigma^2 = V(\theta) < \infty$ then

$$7) \quad \lim_{t \rightarrow \infty} \frac{d^2(t)}{t} = \frac{\sigma^2}{\lambda^3}$$

Theorem 3 (Asymptotic Normality)

If $\sigma^2 < \infty$ then for all real x

$$8) \quad \lim_{t \rightarrow \infty} P \left\{ \frac{\xi_t - \frac{t}{\lambda}}{\sqrt{t \sigma^2 / \lambda^3}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{u^2}{2}} du$$

Theorem 4 (Blackwell's theorem)

If the inter-arrival time θ is not a lattice random variable and $\lambda = E(\theta) < \infty$ then for any $h > 0$

$$9) \quad \lim_{t \rightarrow \infty} \frac{m(t+h) - m(t)}{h} = \frac{1}{\lambda}$$

Theorem 5 (W.L. Smith) If $g(u)$ has bounded variation in the interval $[0, \infty)$ and $F(x)$ is not a lattice distribution and its mean $\alpha < \infty$, then we have

$$\lim_{t \rightarrow \infty} \int_0^t g(t-u) dF(u) = \frac{1}{\alpha} \int_0^{\infty} g(u) du$$

We now describe the renewal theoretic approach to the problems of particle counting. Assume that particles arrive at a counter at times $\tau_0 = 0 < \tau_1 < \tau_2 < \tau_3 < \dots < \tau_n < \dots$ where the $\theta_n = \tau_n - \tau_{n-1}$ are independent, identically distributed positive random variables. Since most counters have a positive "resolving time" not all the particles that arrive are counted. Let the subsequence of $\{\tau_n\}$ denoting the arrival times of the particles actually registered be $0 = \tau'_0 < \tau'_1 < \tau'_2 < \dots$. Again the $\theta'_n = \tau'_n - \tau'_{n-1}$ the times between successive registrations are identically distributed positive random variables. Therefore the primary renewal process $\{\theta_n, F_t\}$ generates a secondary renewal process $\{\theta'_n, \nu'_t\}$ where ν'_t = the number of particles counted in time $(0, t]$. We denote by $R(x)$ the probability $P[\theta'_n \leq x]$.

In order to ascertain the asymptotic behavior of the number of recorded particles $\nu(t)$, it suffices to determine the common mean m and variance σ^2 of the distribution function $R(x)$. Once we know these values, then, by the basic renewal theorems given earlier,

$$\lim_{t \rightarrow \infty} \frac{E(\nu(t))}{t} = \frac{1}{m} \quad \lim_{t \rightarrow \infty} \frac{\text{Var}[\nu(t)]}{t} = \frac{\sigma^2}{m^3}$$

The mechanism of the counter used determines how the subsequence $\{\tau'_n\}$ of "recorded events" is selected from the primary sequence of events

$\{\tau_n\}$. We say that the counter is free at time t , if it is in condition to register a particle arriving at that time, otherwise the counter is said to be locked. The two standard counter mechanism are the type I and type II counters. In a type I counter, if a particle arrives when the counter is free then the counter is locked for a random time χ called the dead time or holding time or impulse time of the counter. Particles arriving during the time χ are not counted and have no effect on the counter's operation. The type II counter differs from the type I counter in that every arriving particle locks the counter for a random locking time χ . Therefore a particle now is registered if and only if at the time of its arrival all the dead times produced by the previous arrivals have expired. With a type I counter a particle is registered if at the time of its arrivals all the dead time produced by the previous registered particles have expired. It is assumed, of course, that the dead times produced by the arriving particles (type II counter) or by the registered particles (type I counter) are identically distributed independent random variables.

When the particles arrive according to a Poisson process, i.e., $F(x) = P[\Theta_n \leq x] = 1 - e^{-\lambda x}$ the following results are known, (see Takács [5] and Smith [4]).

For a type I counter
$$m = \frac{1 + \lambda E[\chi]}{\lambda} \quad \sigma^2 = \frac{1 + \lambda^2 V(\chi)}{\lambda^2}$$

where χ is the dead time random variable $P[\chi \leq x] = H(x)$

Thus

$$\lim_{t \rightarrow \infty} \frac{E[\nu(t)]}{t} = \frac{\lambda}{1 + \lambda E[\chi]} \quad \lim_{t \rightarrow \infty} \frac{V[\nu(t)]}{t} = \lambda \frac{1 + \lambda^2 V(\chi)}{1 + E(\chi)}$$

For a type II counter:

$$m = \frac{1}{\lambda} e^{\lambda E[\chi]}$$

while

$$\sigma^2 = \frac{2e\lambda E[X]}{\lambda} \int_0^\infty \exp[\lambda \int_t^\infty 1 - H(y) dy - 1] dt + \frac{2e\lambda E[X]}{\lambda^2} \frac{2 E[X]}{\lambda^2}$$

In order to present our shorter proof of Pyke's and Malinquist's formulas for the type I counter we must give one more definition.

Definition: If we have the first registration at $\tau_0 = 0$, an event of the primary sequence is registerable if it arrives after the dead time λ_0 produced by the first particle.

Theorem 6 (Pyke's Formula): If $P[\tau_n - \tau_{n-1} \leq x] = F(x)$ and the $\theta_n = \{\tau_n - \tau_{n-1}\}$ are mutually independent and if the dead times λ_n produced by the n -th particle (if it is registered) are distributed as $H(x)$ and the $\{\lambda_n\}$ are mutually independent and also are independent of the sequence $\{\tau_n\}$. Then $\theta_n' = \tau_n' - \tau_{n-1}'$ has the distribution

$$P[\theta_n' \leq z] = R(z) = \int_0^z H(u-) [1 - F(z-u)] dm(u)$$

where

$$m(x) = \sum_{n=1}^{\infty} F_n(x).$$

Proof: Clearly we can assume that there is a registration at time 0 i.e. $\tau_0 = \tau_0' = 0$. Then

$$\begin{aligned} R(z) &= P[\text{there is a registration in } (0, z)] \\ &= P[\text{there is a registerable event in } (0, z)] \\ &= P[\text{the last event in } (0, z] \text{ is registerable}] \end{aligned}$$

$$(11) \quad = \sum_{n=1}^{\infty} \int_0^z H(u-) \{1 - F(z-u)\} dF_n(u).$$

$$(12) \quad = \int_0^z H(u-) [1 - F(z-u)] dm(u).$$

The reasoning is as follows: the probability that the n -th event is the last one occurring in $(0, z]$ and is registerable is by the total probability theorem $\int_0^z H(u-) [1 - F(z-u)] dF_n(u)$ (because the n -th event occurs at time u , $0 < u \leq z$). As the last event in $(0, z]$ can be the n -th for $n = 0, 1, 2, \dots$ again by the total probability theorem we obtain that

$$\begin{aligned} R(z) &= \sum_{n=1}^{\infty} \int_0^z H(u-) \{1 - F(z-u)\} dF_n(u) \\ &= \int_0^z H(u-) [1 - F(z-u)] dM(u) \end{aligned}$$

Theorem 7. (Malmquist's Formula). Under the same general assumptions of Pyke's theorem, except that now we assume that the counter is locked for a constant time d when it registers a particle, we have

$$(3) \quad R(z) = \begin{cases} 0 & z < d \\ F(z) - F(d) + \int_0^d \{F(z-x) - F(d-x)\} dM(x) & . \end{cases}$$

Proof:

$$R(z) = P[\text{there is a registerable event in } (0, z]]$$

$$= P[\text{there is a first registerable event in } (0, z)]$$

Here the first registerable event is the first one to arrive after time d . The first arrival is the first registerable if it arrives in $(d, z]$. The probability of this is just $F(z) - F(d)$. If $n \geq 2$, the n -th particle is the first registerable if the $(n-1)$ -st particle arrives at time x , $x \leq d$ and the n -th arrives between d and z . By the total probability theorem the probability that the n -th ($n \geq 2$) is first registerable is therefore

$$\int_0^d \{F(z-x) - F(d-x)\} dF_{n-1}(x)$$

Applying the total probability theorem again (on n) we obtain

$$\begin{aligned} R(z) &= F(z) - F(d) + \sum_{n=2}^{\infty} \int_0^d \{F(z-x) - F(d-x)\} dF_{n-1}(x) \\ &= F(z) - F(d) + \int_0^d \{F(t-x) - F(d-x)\} dM(x) . \end{aligned}$$

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Chapter II

$$G/E_2/\infty$$

Introduction:

Consider a type II counter. The particles arrive at the instants τ_1, τ_2, \dots where $\{\tau_n - \tau_{n-1}\} \quad n = 1, 2, \dots$ are identically distributed positive random variables with distribution function $F(x)$. Let

$$1) \quad \varphi(s) = \int_0^{\infty} e^{-sx} dF(x)$$

$$2) \quad \alpha = \int_0^{\infty} x dF(x)$$

$$3) \quad \sigma^2 = \int_0^{\infty} (x-\alpha)^2 dF(x)$$

We shall assume that $F(x)$ is not a lattice distribution and that $\alpha < \infty$. Recall that in a type II counter every arriving particle produces an impulse but only those particles arriving when the counter is free will be registered. Let λ_n denote the impulse time or dead time produced by the n -th arriving particle. We suppose that the $\{\lambda_n\} \quad n = 1, 2, \dots$ are identically distributed mutually independent positive random variables which are also independent of $\{\tau_n\}$. In the present chapter we suppose that the λ_n have an Erlang-2 distribution.

$$4) \quad H(x) = P\{\lambda_n \leq x\} = 1 - e^{-\mu x} - \mu x e^{-\mu x}$$

and we shall derive a formula for the Laplace transform of the r -th binomial moment of the number of impulses present in the counter at time t . This random variable will be called $\eta(t)$. Although theoretically we can invert the binomial moments and obtain the exact distribution of $\eta(t)$ the formulas obtained are too complicated for practical use in this manner.

The random variable $\eta(t)$ is not very interesting from the point of view of particle counting but is the object of main interest in the theory of infinitely many server queueing systems since $\eta(t)$ is now the queue size of the system $G/E_2/\infty$.

Notation: $P_k(t) = P[\eta(t) = k]$

The r -th binomial moment of $\eta(t)$ is given by

$$B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t)$$

By Jordan's inversion formula

$$P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r(t)$$

$$B_r(s) = \int_0^{\infty} e^{-st} B_r(t) dt \quad \operatorname{Re}(s) > 0$$

is the Laplace transform of the r -th binomial moment of $\eta(t)$.

$$= \int_0^{\infty} x dH(x) = \int_0^{\infty} [1 - H(x)] dx \quad \text{will always be}$$

assumed to exist and be finite for all dead time distributions used in this paper.

§ 2. A Review of the Results of Takács

In this section we shall assume that $\eta(0) = 0$ and $P[\tau_1 \leq x] = F(x)$. Consider the generating function

$$(2.1) \quad G(t, z) = \sum_{k=0}^{\infty} P_k(t) z^k.$$

Theorem 1: The generating function $G(t, z)$ satisfies the following

integral equation:

$$2.2. \quad G(t, z) = [1 - F(t)] + \int_0^t G(t-x, z) \{z + (1-z) H(t-x)\} dF(x)$$

Proof:

By the theorem of total probability

$$P_0(t) = 1 - F(t) + \int_0^t H(t-x) P_0(t-x) dF(x)$$

since we have no impulses present in the machine if either there is no arrival up to time t (the probability of this event is $1 - F(t)$) or there is a first particle arriving at x , $0 < x < t$; the impulse it produces expires by time t (the probability of this is just $H(t-x)$) and the process "renewed" at x has no impulses present at time t $[P_0(t-x)]$. Similarly we obtain for $k = 1, 2, \dots$

$$(2.4) \quad P_k(t) = \int_0^t \{P_k(t-x) H(t-x) + P_{k-1}(t-x) [1 - H(t-x)]\} dF(x)$$

Multiplying the equations (2.4) by z^k and adding over $k = 0, 1, 2, \dots$ we obtain (2.2) as desired.

Theorem 2. The binomial moments $B_r(t)$ exist for all t and can be determined from the following recurrence formulas:

$$B_0(t) = 1 \text{ and}$$

$$(2.5) \quad B_r(t) = \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dm(x) \quad r = 1, 2, \dots$$

where $m(x) = \sum_{n=1}^{\infty} F_n(x)$ and $F_n(x)$ denotes the n -th - fold convolution of $F(x)$ with itself. Further

$$(2.6) \quad P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r(t)$$

Proof: Clearly $B_r(t) = \frac{1}{r!} \left[\frac{d^r G(t, z)}{dz^r} \right]_{z=1}$ $r = 0, 1, 2, \dots$

Since $B_0(t) = 1$ upon differentiating equation (2.2) r times with respect to z and evaluating at $z = 1$ we obtain

$$B_r(t) = \int_0^t B_r(t-x) dF(x) + \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dF(x)$$

This a linear integral equation of the Volterra type for the unknown $B_r(t)$. The solution is well known (it is obtainable by taking Laplace transforms) to be

$$B_r(t) = \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dm(x).$$

To prove (2.6) we must show that the generating function $G(t, z)$ is analytic at $z = 1$. It suffices to show that $B_r(t) \leq \frac{C^r}{r!}$ for some constant C . By (2.5) we can write

$$B_r(t) = \int_{t_1+t_2+\dots+t_r \leq t} \dots \int [1 - H(t-t_1)] \dots [1 - H(t-t_r)] dm(t_1) \dots dm(t_r)$$

Let h be a fixed positive number and let $k(x) = H(x-h)$. Since

$$m(t+h) - m(t) \leq 1 + m(h) \quad \text{for all } t \geq 0 \text{ we}$$

easily obtain

$$B_r(t) \leq \left(\frac{1+m(h)}{h} \right)^r \int_{x_1+\dots+x_r \leq t+h} [1 - k(x_1)] \dots [1 - k(x_r)] dx_1 \dots dx_r.$$

$$= \left(\frac{1+m(h)}{h} \right)^r \frac{\left\{ \int_0^{t+h} [1 - k(x)] dx \right\}^r}{r!}$$

$$\leq \left[\frac{1+m(h)}{h} \right]^r \frac{[h+\rho]^r}{r!} \quad \text{where } \rho = \int_0^\infty x dH(x) < \infty$$

thus setting $C = \frac{1+m(h)}{h} \cdot (h+\rho)$ we have proved that

$G(t, z)$ is analytic at $z = 1$ and thus it is permissible to invert

$$B_r(t) = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t) \quad \text{and we obtain}$$

$$P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r(t).$$

Before proceeding to discuss the limiting behavior we review some results from the theory of functions of bounded variation.

Lemma 1: The product of two functions each of which is of bounded variation is also of bounded variation.

Lemma 2: The integral of a function of bounded variation on a finite interval $[a, b]$ is also of bounded variation on that interval.

Theorem 3: If $\rho = \int_0^\infty x dH(x) < \infty$, $\lambda = \int_0^\infty x dF(x)$ and $F(x)$ is not a lattice distribution then the limiting distribution

$$\lim_{t \rightarrow \infty} P(\eta(t) = k) = P_k^* \quad k = 0, 1, 2, \dots \text{ exists}$$

and is independent of the initial distribution and we have

$$(2.7) \quad P_k^* = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r^*$$

where B_r^* is the r -th binomial moment of $\{P_k^*\}$ and can be determined in the following way:

$$(2.8) \quad B_0^* = 1 \quad \text{and} \quad B_r^* = \frac{1}{\alpha} \int_0^\infty B_{r-1}(t) [1 - H(t)] dt$$

Proof:

First we shall prove the theorem in the particular case that $\eta(0) = 0$.

As $B_0(t) = 1$ clearly $B_0^* = 1$ and by (2.5) we have

$$B_1(t) = \int_0^t [1 - H(t-x)] dm(x)$$

As $1 - H(x)$ is monotone non-increasing we may apply Smith's theorem to deduce that B_1^* exists and equals

$$B_1^* = \frac{1}{\alpha} \int_0^\infty [1 - H(t)] dt = 1/\alpha.$$

By lemma 1 $B_1(t)$ is of bounded variation on every finite interval and as the limit B_1^* exists $B_1(t)$ is bounded on the entire line. We can show (2.8) by induction. If we assume that

$$\lim_{t \rightarrow \infty} B_{r-1}(t) = B_{r-1}^* \text{ exists, then by Smith's theorem}$$

$$\text{applied to} \quad B_r(t) = \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dm(x)$$

we can conclude that

$$\lim_{t \rightarrow \infty} B_r(t) = B_r^* \text{ exists and is given by (2.8) .}$$

$$\text{Since} \quad B_r^*(t) \leq \frac{G^r}{r!} \quad \text{for} \quad t \geq 0$$

$$(2.9) \quad \lim_{t \rightarrow \infty} G(t, z) = G^*(t) \quad \text{exists and we have}$$

$$(2.10) \quad G^*(z) = \sum_{r=0}^{\infty} B_r^* (z-1)^r.$$

The series (2.10) is convergent for all z . By (2.8) we see that

$$(2.11) \quad G^*(z) = 1 - \frac{(1-z)}{\lambda} \int_0^{\infty} G(t, z) [1 - H(t)] dt.$$

Now $G^*(1) = 1$ and according to the continuity theorem for generating functions it follows that the limiting probabilities

$$\lim_{t \rightarrow \infty} P[\eta(t) = k] = P_k^* \quad k = 0, 1, 2, \dots$$

exist and that

$$G^*(z) = \sum_{k=0}^{\infty} P_k^* z^k.$$

Finally from (2.10)

$$P_k^* = \frac{1}{k!} \left. \frac{d^k G^*(z)}{dz^k} \right|_{z=0} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r^*$$

and thus the B_r^* ($r = 0, 1, 2, \dots$)

are indeed, the binomial moments of the distribution $\{P_k^*\}$. This completes the proof for the initial state 0. If we consider an arbitrary initial state then the only difference is that $\eta(t)$ is replaced by

$\tilde{\eta}(t - \tau_1) + \epsilon(t)$ where τ_1 is the random arrival time of the first new customer, $\tilde{\eta}(t)$ has the same distribution as our $\eta(t)$ and

$\lim_{t \rightarrow \infty} P[\epsilon(t) = 0] = 1$. ($\epsilon(t)$ is the number of the original customer

remaining in the queue at time t . Clearly $\int_0^{\infty} x dH(x) < \infty$ implies that

$\epsilon(t) \rightarrow 0$ as $t \rightarrow \infty$). Consequently, $\tilde{\eta}(t - \tau_1) + \epsilon(t)$ has the same asymptotic distribution as the special case $\eta(0) = 0$ that we considered.

Thus the proof of the main theorem is now complete.

§ 3. The Explicit Solution in an Operational Form

The fundamental recurrence between $B_r(t)$ and $B_{r-1}(t)$ is

$$B_r(t) = \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dm(x) .$$

When $H(x)$ is Erlang-2 this becomes

$$(3.1) \quad B_r(t) = \int_0^t B_{r-1}(t-x) [e^{-\mu(t-x)} + \mu(t-x) e^{-\mu(t-x)}] dm(x)$$

Upon taking Laplace transforms we obtain the recurrence equation for the corresponding transforms:

$$(3.2) \quad B_r(s) = \frac{\phi(s)}{1-\phi(s)} [B_{r-1}(s+\mu) - \mu B_{r-1}'(s+\mu)] .$$

As the units in which we measure time are completely arbitrary we may assume that $\mu = 1$ without loss of generality. In order to make the discussion as simple as possible we introduce the following notations and operations.

$$d(s) = \frac{\phi(s)}{1-\phi(s)} .$$

E: shifts $r \rightarrow r+1$ e.g. $B_1(s) = E\beta_0(s)$; $\beta_r(s) = E^r\beta_0(s)$

F: shifts the argument $s \rightarrow s+1$ e.g. $Fd(s) = d(s+1)$; $F^k d(s) = d(s+k)$

D: is the differentiation operator.

It is important to note that while D, E and F commute with each other, they do not commute with functions of s . Our equation (3.2) now becomes the operational equation:

$$(3.2') \quad E^r \beta_0(s) = d(1-D)FE^{r-1} \beta_0(s) \quad \text{given } \beta_0(s) = \frac{1}{s} .$$

Theorem 3.1

$$(3.3) \quad E^r = \sum c(a_1, \dots, a_r) d D^{a_1} F d D^{a_2} F^2 d \dots D^{a_{r-1}} F^{r-1} d (1-D)^{a_r} F^r$$

$$(a_1, \dots, a_{r-1}, a_r) \geq 0$$

$$a_r \geq 1 \quad \sum a_i = r$$

$$\sum_{i=1}^{j-1} a_i \leq j \quad \text{for all } j \leq r-1$$

where

$$(3.4) \quad c(a_1, \dots, a_r) = (-1)^{\sum_{i=1}^{r-1} a_i} \prod_{j=1}^{r-1} \binom{j - \sum_{i=1}^{j-1} a_i}{a_j}$$

Before giving the proof we give some preliminary explanation. To find $E^r(1/3) = \beta_r(s)$ we must first enumerate all the r -tuples (a_1, \dots, a_r) such that 1) $a_r \geq 1$ 2) $\sum_{i=1}^{j-1} a_i \leq j$ for $j \leq r-1$ 3) $\sum_{i=1}^r a_i = r$

For each r -tuple consider

$$(3.5) \quad c(a_1, \dots, a_r) \cdot d(s) D^{a_1} d(s+1) D^{a_2} d(s+2) \dots D^{a_{r-1}} d(s+(r-1)) (1-D)^{a_r} d(s+r)$$

where $c(a_1, \dots, a_r)$ is a constant depending on the arrangement chosen and $(1-D)^{a_r}$ is to be expanded in the binomial formula. $\beta_r(s)$ is equal to the sum of expressions of the form of (3.5).

It is interesting to identify the combinatorial meaning of the terms in the product of the coefficient $c(a_1, \dots, a_r)$. For choosing a_1 we are given one ball j we can take it (take 1) or leave it (take 0). For choosing a_2 we are given a new ball; if $a_1 = 0$ we choose a_2 balls from two balls i.e. $a_2 = 0, 1$, or 2 while if $a_1 = 1$ $a_2 = 0$ or 1. Similarly, we choose a_j balls from j balls minus the number of balls

chosen previously, i.e., choose a_j from $j = \sum_{i=1}^{j-1} a_i$. The product

$\prod_{j=1}^{r-1} \binom{j-1}{\sum_{i=1}^{j-1} a_i} a_j$ equals the number of ways we can put r indistinguishable balls into r cells in such a way that the number of balls in the first j cells is always $\leq j$.

Proof of the theorem:

The method of proof is by induction. As the formula holds for

$$E^1 = a(1-D)F \quad \text{and} \quad E^2 = aFa(1-D)^2F^2 + aDfa(1-D)F^2$$

we assume that it is true for E^{r-1} and show it holds for E^r .

$$E^r = a(1-D)F \sum_{a_1, \dots, a_{r-1}, a} c(a_1, \dots, a_{r-1}, a) aD^{a_1} F a \dots D^{a_{r-2}} F^{a_{r-2}} a(1-D)^2 F^{r-2}$$

$$(a_1, \dots, a_{r-2}, a)$$

$$\sum_{i=1}^{r-2} a_i + a = r-1, a \geq 1$$

$$\sum_{i=1}^{j-1} a_i \leq j \quad \text{for all } j \leq r-2$$

To see what is happening look at the terms in E^r to which

$$c(a_1, \dots, a_{r-2}, a) aD^{a_1} F a \dots D^{a_{r-2}} F^{a_{r-2}} a(1-D)^2 F^{r-2} \quad \text{contributes i.e.,}$$

$$\begin{aligned} c(a_1, \dots, a_{r-2}, a) &\longrightarrow + c(0, a_1, a_2, \dots, a_{r-2}, a+1) \\ &- c(1, a_1, \dots, a_{r-2}, a) \\ &+ c(0, a_1+1, a_2, \dots, a_{r-2}, a) \\ &- c(0, a_1, a_2+1, a_3, \dots, a_{r-2}, a) \\ &\vdots \\ &- c(0, a_1, a_2, \dots, a_{r-3}, a_{r-2}+1, a) \end{aligned}$$

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(we omitted the operations since the a 's in the c -term determine the corresponding operation). By inverting the above system and remembering that $a_1 = 1$ or 0 we see that

$$(3.6) \quad c(1, a_2, \dots, a_{r-1}, a) = -c(a_2, \dots, a_{r-1}, a)$$

while

$$(3.7) \quad c(0, a_2, \dots, a_{r-1}, a) = +c(a_2, a_3, \dots, a_{r-1}, a-1) \\ - c(a_2-1, a_3, \dots, a) \\ - c(a_2, a_3-1, a_4, \dots, a) \\ \vdots \\ - c(a_2, a_3, \dots, a_{r-2}, a_{r-1}-1, a)$$

Notice that if $a_1 = 2$ $c(a_2, a_3-1, \dots)$ will be zero as $\binom{1}{2} = 0$ and in general $\binom{n}{k} = 0$ if $k > n$. With this convention it is easy to see that (3.6) holds i.e.

$$c(1, a_2, \dots, a_{r-1}, a) = -c(a_2, \dots, a_{r-1}, a-1) \quad \text{since}$$

$$(-1)^{\sum a_i} \binom{1}{a_2} \binom{2-a_2}{a_3} \dots \binom{r-2-\sum_{i=2}^{r-2} a_i}{a_{r-1}} = (-1)^{\{\sum_{i=1}^{r-1} a_i + 1\}} \cdot \{\text{same product}\}$$

To see that (3.7) holds for our formula we use the elementary relation

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad . \quad \text{Thus :}$$

$$c(0, a_2, \dots, a_{r-1}, a) = (-1)^{\sum_{i=1}^{r-1} a_i} \prod_{j=2}^{r-1} \binom{j - \sum_{i=1}^{j-1} a_i}{a_j}$$

$$= (-1)^{\sum_{i=1}^{r-1} a_i} \binom{2}{a_2} \binom{3 - a_2 - a_1}{a_2} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-2} a_i}{a_{r-1}}$$

as $\binom{2}{a_2} = \binom{1}{a_2} + \binom{1}{a_2-1}$ this becomes

$$\binom{1}{a_2-1} \binom{3 - a_2 - a_1}{a_3} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-1} a_i}{a_{r-1}}$$

$$+ \binom{1}{a_2} \binom{3 - a_2 - a_1}{a_3} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-1} a_i}{a_{r-1}}$$

Again use $\binom{3 - a_2 - a_1}{a_3} = \binom{2 - a_2 - a_1}{a_3 - 1} + \binom{2 - a_2 - a_1}{a_3}$

and our expression becomes

$$(-1)^{\sum_{i=1}^{r-1} a_i} \binom{1}{a_2 - 1} \binom{3 - a_2 - a_1}{a_3} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-1} a_i}{a_{r-1}}$$

$$(-1)^{\sum_{i=1}^{r-1} a_i} \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3 - 1} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-1} a_i}{a_{r-1}}$$

$$(-1)^{\sum_{i=1}^{r-1} a_i} \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \binom{4 - a_3 - a_2 - a_1}{a_4} \dots \binom{r - 1 - \sum_{i=1}^{r-1} a_i}{a_{r-1}}$$

Proceeding in this way (i.e., decompose a_4 term next etc.) we obtain

$$\begin{aligned} c(a_1, \dots, a_{r-1}, a) = & (-1)^{\sum_{i=1}^{r-1} a_i} \text{ times } \binom{1}{a_2 - 1} \binom{3 - a_2 - a_1}{a_3} \binom{4 - \sum_{i=1}^3 a_i}{a_4} \dots \binom{r-1 - \sum_{i=1}^{r-2} a_i}{a_{r-1}} \\ & + \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \binom{4 - \sum_{i=1}^3 a_i}{a_4} \dots \binom{r-1 - \sum_{i=1}^{r-2} a_i}{a_{r-1}} \\ & + \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \binom{3 - \sum_{i=1}^3 a_i}{a_4} \binom{5 - \sum_{i=1}^4 a_i}{a_5} \dots \binom{r-1 - \sum_{i=1}^{r-2} a_i}{a_{r-1}} \\ & \vdots \\ & + \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \dots \binom{r-3 - \sum_{i=1}^{r-3} a_i}{a_{r-2}} \binom{r-1 - \sum_{i=1}^{r-2} a_i}{a_{r-1}} . \end{aligned}$$

This last line in the above system becomes

$$\begin{aligned} & \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \dots \binom{r - 3 - \sum_{i=1}^{r-3} a_i}{a_{r-2}} \binom{r - 2 - \sum_{i=1}^{r-2} a_i}{a_{r-1} - 1} \\ & + \binom{1}{a_2} \binom{2 - a_2 - a_1}{a_3} \dots \binom{r - 3 - \sum_{i=1}^{r-2} a_i}{a_{r-2}} \binom{r - 2 - \sum_{i=1}^{r-2} a_i}{a_{r-1}} \end{aligned}$$

Hence

$$\begin{aligned}
 c(0, a_2, \dots, a_{r-1}, a) &= -c(a_2 - 1, a_3, \dots, a) \\
 &= -c(a_2, a_3 - 1, \dots, a) \\
 &\vdots \\
 &= -c(a_2, a_3, \dots, a_{r-1} - 1, a) \\
 &= -c(a_2, a_3, \dots, a_{r-1}, a - 1)
 \end{aligned}$$

as desired and we've shown that the c 's fit the relations (3.6) and (3.7) thus if (3.3) holds for E^{r-1} , it holds for E^r . Therefore, as (3.3) is correct for $r = 1, 2$, by the induction hypothesis E^r is given by (3.3) for all r .

4. The Limiting Distribution of $\eta(t)$.

By standard Tauberian theory (see e.g. Doetsch [5]) if $\lim_{t \rightarrow \infty} B_r(t)$ exists then $\lim_{t \rightarrow \infty} B_r(t) = \lim_{s \rightarrow 0} s\beta_r(s)$. Since we know (§2) that

$B_r^* = \lim_{t \rightarrow \infty} B_r(t)$ exists and that $\{B_r^*\}$ for $r = 0, 1, 2, \dots$ are the binomial moments of the limiting distribution $P_k^* = \lim_{t \rightarrow \infty} P[\eta(t) = k]$

$k = 0, 1, 2, \dots$ we can apply the Tauberian theorem referred to. We now compute B_1^* , B_2^* and B_3^* .

Lemma 1: $\lim_{s \rightarrow 0} \frac{s\varphi(s)}{1-\varphi(s)} = \frac{1}{\lambda}$ where $\lambda = \int_0^\infty x dF(x) < \infty$

Proof:
$$\frac{\varphi(s)}{1-\varphi(s)} = \frac{1 - \lambda s + O(s^2)}{\lambda s + O(s^2)}$$

$$\frac{s\varphi(s)}{1-\varphi(s)} = \frac{1 - O(s)}{\lambda + O(s)}$$

$$\lim_{s \rightarrow 0} \frac{s \varphi(s)}{1 - \varphi(s)} = \frac{1}{\lambda}$$

The first three binomial moments of the limiting distribution are

$$1) \quad B_1^* = \frac{2}{\lambda}$$

$$\text{Proof:} \quad \lim_{s \rightarrow 0} s \beta_1(s) = \lim_{s \rightarrow 0} \frac{s \varphi(s)}{1 - \varphi(s)} \cdot \frac{s+2}{(s+1)^2} = \frac{2}{\lambda}$$

$$2) \quad B_2^* = \frac{1}{\lambda} \left\{ \frac{5}{4} \frac{\varphi(1)}{1 - \varphi(1)} - \frac{3}{4} \left[\frac{\varphi}{1 - \varphi} \right]'(1) \right\}$$

$$\text{Proof:} \quad B_2(s) = dF d(1-D)^2 F^2(1/s) - dDF d(1-D) F^2 1/s$$

$$\begin{aligned} B_2(s) &= \frac{\varphi(s)}{1 - \varphi(s)} \cdot \frac{\varphi(s)}{1 - \varphi(s+1)} [1 - 2D + D^2] \frac{1}{s+2} - \frac{\varphi(s)}{1 - \varphi(s)} \left[\frac{\varphi}{1 - \varphi} \right]'(s+1) [1-D] \frac{1}{s+2} \\ &= \frac{\varphi(s)}{1 - \varphi(s)} \cdot \frac{\varphi(s+1)}{1 - \varphi(s+1)} \left[\frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{2}{(s+2)^2} \right] - \frac{\varphi(s)}{1 - \varphi(s)} \left(\frac{\varphi}{1 - \varphi} \right)' \left[\frac{1}{s+2} - \frac{1}{(s+2)^2} \right] \end{aligned}$$

therefore

$$\lim_{s \rightarrow 0} s \beta_2(s) = \frac{1}{\lambda} \left[\frac{5}{4} \frac{\varphi(1)}{1 - \varphi(1)} - \frac{3}{4} \left(\frac{\varphi}{1 - \varphi} \right)'(1) \right].$$

Similarly it can be shown that

$$\begin{aligned}
 3) \quad B_3^* &= \frac{1}{2} \left[\frac{\varrho(1)}{1-\varrho(1)} \cdot \frac{\varrho(2)}{1-\varrho(2)} \cdot \frac{26}{27} - \left(\frac{\varrho}{1-\varrho} \right)^1 (1) \frac{\varrho(2)}{1-\varrho(2)} \cdot \frac{17}{27} \right] \\
 &+ \frac{34}{272} \left[\left(\frac{\varrho}{1-\varrho} \right)^1 (1) \left(\frac{\varrho}{1-\varrho} \right)^1 (2) \right] \\
 &+ \frac{4}{92} \left[\left(\frac{\varrho}{1-\varrho} \right)^1 (1) \left(\frac{\varrho}{1-\varrho} \right)^1 (2) + \frac{\varrho(1)}{1-\varrho(1)} \cdot \left(\frac{\varrho}{1-\varrho} \right)^{(2)} (2) \right]
 \end{aligned}$$

In the special case of Poisson arrivals $\varrho(s) = \lambda/\lambda+s$ and we obtain

$$B_1^* = 2\lambda \quad B_2^* = 2\lambda^2 \quad B_3^* = \frac{(2\lambda)^3}{3!}$$

which agree with earlier results of Takács who showed that

$$P_k^* = \lim_{t \rightarrow \infty} P_k(t) = e^{-\lambda\rho} \frac{(\lambda\rho)^k}{k!} \quad \text{where } \rho = \text{mean impulse time} = 2$$

if the locking time has an Erlang-2 distribution. Since the binomial moments of the Poisson distribution of density ν are given by $B_r = \nu^r/r!$, our formulas coincide with those of Takács.

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Chapter III

 $G/\max-m/\infty$ 1 Introduction

In this chapter we again determine the Laplace transforms of the binomial moments of $\eta(t)$ = the number of impulses present in the counter at time t = the queue size of the queueing system $G/\max-m/\infty$. Thus we suppose that particles arrive at a type II counter according to a recurrent process where the inter-arrival times $\{\tau_n - \tau_{n-1}\}$ $n = 1, 2, \dots$ are independent positive, non-lattice random variables.

Let $F(x) = P[\tau_n - \tau_{n-1} \leq x]$

$$\alpha = \int_0^{\infty} x dF(x) < \infty$$

$$\psi(s) = \int_0^{\infty} e^{-sx} dF(x)$$

Finally we assume that the locking time χ_n produced by the n -th particle is the maximum of m independent identically distributed exponential random variables i.e.,

$$H(x) = P[\chi_n \leq x] = [1 - e^{-\mu x}]^m = \sum_{j=0}^m (-1)^j \binom{m}{j} e^{-\mu x j}$$

It will be seen that the recurrence equations for the Laplace transforms of the binomial moments will be easier to solve for the general $\max-m$ distribution than for the case of Erlang-2 dead time distribution dealt with in the previous chapter.

2. The Transient Behavior of $\gamma(t)$

In 1958 [1] Professor L. Takács derived the following recurrence equations for the r -th binomial moment at time t .

$$(2.1) \quad B_r(t) = \int_0^t B_{r-1}(t-x) [1 - H(t-x)] dm(x)$$

where $H(x)$ is the dead-time distribution. In our situation the equation to be solved becomes

$$(2.2) \quad B_r(t) = \int_0^t B_{r-1}(t-x) \left\{ \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} e^{-\mu(t-x)j} \right\} dm(x)$$

Letting $\beta_r(s) = \int_0^\infty e^{-st} B_r(t) dt$ be the Laplace transform of $B_r(t)$ we obtain from (2.2) the equation

$$(2.3) \quad \beta_r(s) = \frac{\varphi(s)}{1-\varphi(s)} \left\{ \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} \beta_{r-1}(s+j\mu) \right\}$$

for the Laplace transform of the r -th binomial moment. It is to be noticed that no derivative appears in (2.3) in contrast to equation (3.2) of Chapter II and thus we have a pure m -th order difference equation to solve. As $B_0(t) = 1$ for all t $\beta_0(s) = \frac{1}{s}$ for all s . For convenience

we shall set $\mu = 1$, $\frac{\varphi(s)}{1-\varphi(s)} = d(s)$ and $d_k = d(s+k)$. Then (2.3) becomes

$$(2.4) \quad \beta_r(s) = d(s) \left\{ \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} \beta_{r-1}(s+j) \right\}$$

Theorem: the solution of (2.4) is

$$(2.5) \beta_r(s) = \sum_{k=r}^{m} \frac{(-1)^{k+r}}{s+k} \left\{ \sum_{\substack{\text{over all partitions} \\ (u_1, \dots, u_r) \text{ of } k \\ \text{such that} \\ \sum_{j=1}^r u_j = k \quad 1 \leq u_j \leq m}} d_0 d_{u_1} d_{u_1+u_2} \dots d_{u_1+u_2+\dots+u_{r-1}} \prod_{i=1}^r \binom{m}{u_i} \right.$$

Proof:

If $r = 0$ $\beta_0(s) = 1/s$. If $r = 1$ from (2.4)

$$(2.6) \quad \beta_1(s) = d_0 \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} \frac{1}{s+j}.$$

Now there is one and only one partition of j satisfying the conditions of (2.5), namely, j itself. Therefore the coefficient of $\frac{1}{s+j}$ is

$(-1)^{j+1} d_0 \binom{m}{j}$ as desired and thus formula (2.5) gives the solution

if $r = 1$. We proceed by induction. Assume that the theorem is true for all moments up to $r - 1$. In particular

$$\beta_{r-1}(s) = \sum_{k=r-1}^{m(r-1)} \frac{(-1)^{k+r-1}}{s+k} \left\{ \sum_{\substack{\text{over all partitions} \\ (u_1, u_2, \dots, u_{r-1}) \text{ of } k \\ \text{such that} \quad \sum_{i=1}^{r-1} u_i = k \\ \text{and } 1 \leq u_i \leq m}} d_0 d_{u_1} d_{u_1+u_2} \dots d_{u_1+u_2+\dots+u_{r-2}} \prod_{i=1}^{r-1} \binom{m}{u_i} \right.$$

Since $\beta_r(s) = d(s) \left\{ \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} \beta_{r-1}(s+j) \right\}$ it is fairly clear that

$\beta_r(s)$ is given by our formula because the shift $s + k \rightarrow s + k + j$ has many partitions, each of which comes from a u_1 chosen from $1, \dots, j$ and another partition u_2, \dots, u_r which is a partition of $k + j - u_1$. The proper multiplier in each case is $\binom{m}{u_1}$ or $\prod_{i=2}^r \binom{m}{u_i}$

Thus, all we have done is to introduce new partitions of the numbers r to m the first of which is j .

In detail:

(2.7)

$$\beta_r(s) = \sum_{j=1}^m (-1)^{j-1} \binom{m}{j} \sum_{k=r-1}^{m(r-1)} \frac{(-1)^{k+r-1}}{s+k+j} \cdot \left\{ \begin{array}{l} \sum d_0 d_j d_{u_1+j} \cdots d_{u_1+u_2+j} \cdots d_{u_1+\dots+u_{r-2}+j} \prod_{i=1}^{r-1} \\ \text{over all partitions} \\ (u_1, \dots, u_{r-1}) \text{ of } k \\ \text{such that} \\ \sum_{i=1}^{r-1} u_i = k \quad 1 \leq u_i \leq m \end{array} \right.$$

$$= \sum_{j=1}^m \sum_{k=r-1}^{m(r-1)} \frac{(-1)^{k+r-j}}{s+k+j} \left\{ \begin{array}{l} \sum d_0 d_j d_{u_1+j} \cdots d_{u_1+u_2+\dots+u_{r-2}+j} \binom{m}{j} \prod_{i=1}^{r-1} \binom{m}{u_i} \\ \text{over all partitions} \\ (u_1, u_2, \dots, u_{r-1}) \text{ of } k \\ \text{such that} \\ \sum_{i=1}^{r-1} u_i = k \quad 1 \leq u_i \leq m \end{array} \right.$$

Let $\gamma = k + j$, then γ varies from r to mr and our sum becomes

$$\sum_{\gamma=r}^{mr} \frac{(-1)^{\gamma+r}}{s+\gamma} \left\{ \sum_{\substack{\text{over all partitions of } \gamma \\ \text{such that} \\ \sum_{i=1}^r u_i = \gamma \quad 1 \leq u_i \leq m}} d_0 d_{u_1} d_{u_1+2} \cdots d_{u_1+\dots+u_{r-1}} \prod_{i=1}^r \binom{m}{u_i} \right\}$$

as desired. In words, the j of the first sum (2.7) becomes the first element u_1 of the partition of γ into r parts.

§ 3. The Ergodic Behavior of $\eta(t)$.

Although the formulas are rather cumbersome we can determine the first few moments of the limiting distribution and can use the standard Tauberian theory to obtain a general formula B_r^* the r -th binomial moment of the limiting distribution $\{P_k^*\} = \lim_{t \rightarrow \infty} P[\eta(t) = k]$ (see chapter II § 4). Since B_r^* exists it is given by $B_r^* = \lim_{s \rightarrow 0} s \beta_r(s)$. Hence

$$(3.1) \quad B_r^* = \frac{1}{r} \sum_{k=r}^{mr} \frac{(-1)^{k+r}}{k} \sum_{\substack{\text{all partitions} \\ (u_1, u_2, \dots, u_r) \text{ of } k \\ \text{such that } \sum_{i=1}^r u_i = k \\ 1 \leq u_i \leq m}} \delta_{u_1} \delta_{u_1+u_2} \cdots \delta_{u_1+u_2+\dots+u_{r-1}} \prod_{i=1}^r \binom{m}{u_i}$$

where $\delta_k = \frac{e(k)}{1-e(k)}$.

To illustrate the use of (3.1) we compute B_2^* for the case where $H(x)$ is max 2.

$$B_2^* = \frac{1}{\lambda} \sum_{k=2}^{\infty} \frac{(-1)^{k+2}}{k} \left\{ \sum_{\substack{\text{over all partitions} \\ (u_1, u_2) \text{ of } k \\ \text{such that} \\ u_1 + u_2 = k \quad 1 \leq u_1 \leq 2}} \delta_{u_1} \delta_{u_1+u_2} \binom{m}{u_1} \binom{m}{u_2} \right\}$$

the partitions (u_1, u_2) of $k=2$ are $u_1=1, u_2=1$

" " " " $k=3$ " $u_1=1, u_2=2$ $u_1=2, u_2=1$

" " " " $k=4$ " $u_1=2, u_2=2$

$$\begin{aligned} \text{therefore } B_2^* &= \frac{1}{\lambda} \left\{ \frac{\binom{2}{1}\binom{2}{1}}{2} \delta_1 - \frac{\binom{2}{1}\binom{2}{2}}{3} \delta_1 - \frac{\binom{2}{2}\binom{2}{1}}{3} \delta_2 + \frac{\binom{2}{2}\binom{2}{2}}{4} \delta_2 \right\} \\ &= \frac{1}{\lambda} \left[\frac{4}{3} \delta_1 - \frac{5}{12} \delta_2 \right] . \end{aligned}$$

If $e(s) = \lambda/\lambda+s$, that is if the arrivals form a Poisson Process then

$\delta_k = \lambda/k$ and we obtain

$$B_2^* = \lambda \left[\frac{4}{3} \lambda - \frac{5}{12} \lambda \right] = \frac{(\frac{3}{2} \lambda)^2}{2!} \quad \text{as desired .}$$

Of course $B_1^* = \rho/\lambda$ where ρ is mean "locking time" and λ is mean service time for general distributions.

References: Same as Chapter II.

Chapter IV : A related Batch-Arrival Queueing problem and the determination of the mean time between consecutive registrations.

§ 1 Introduction: So far we have concentrated our attention on the random variable $\eta(t)$ = the number of impulses present in the machine at time t . Although the queue size $\eta(t)$, is of fundamental importance in the theory of infinitely many server queues it is not usually of primary interest in the theory of particle counters. As we remarked in the Introduction, the random variable central to the theory of Counters is the "time between successive registrations" which we denoted by $R(x)$.

In order to ascertain the mean of $R(x)$ when the dead time has a max-m distribution we introduce a new Stochastic Process which may also be of interest in the theory of bulk queues. We shall assume that particles arrive in batches of size m at the type II counter each particle producing an exponentially distributed impulse independently of the other particles. Let $\delta(t)$ be the number of impulses present in the counter in this new process. In general, $\delta(t)$ is not equal to $\eta(t)$, but $\delta(t)$ and $\eta(t)$ will be zero simultaneously. Thus the time between two arrivals finding the counter free (i.e., $R(x)$) will be the same in both cases. We shall see that it is relatively easy to find the mean of $R(x)$ in the $\delta(t)$ process because we can utilize a 1-dimensional imbedded Markov Chain.

§ 2. The General theory for the Bulk-Arrival Model

In this section we shall derive the recurrence equations for the batch arrival system where the particles arrive at the type II counter

(or customers at an infinitely many server queueing system) in batches of size m according to a recurrent process. The inter-arrival distribution $F(x)$ is assumed to be non-lattice and positive. The "dead time" produced by any particle (or service time of a customer) is assumed to have the positive distribution function $H(x)$. Let $\{\tau_n\}$ be the sequence of arrival points of the groups of particles and let λ be the dead time produced by a single particle. We assume that $P[\tau_n - \tau_{n-1} \leq x] = F(x)$ for all n and that the $\{\tau_n - \tau_{n-1}\}$ are mutually independent. Also the individual dead times produced are mutually independent and also independent of the $\{\tau_n\}$ system. Let $\delta(t)$ denote the number of impulses present in the counter at time t (equivalently $\delta(t)$ denotes the queue size at time t). For simplicity we assume that $\delta(0) = 0$ and $P[\tau_1 \leq x] = F(x)$. If we let $P_k(t) = P[\delta(t) = k]$ we can derive an integral equation satisfied by

$$G(t, z) = \sum_{k=0}^{\infty} P_k(t) z^k.$$

Theorem 1: The generating function $G(t, z)$ satisfies the following integral equation

$$(2.1) \quad G(t, z) = [1 - F(t)] + \int_0^t G(t-x, z) [z + (1-z) H(t-x)]^m dF(x)$$

Proof: By the theorem of total probability

$$(2.2) \quad P_0(t) = [1 - F(t)] + \int_0^t [H(t-x)]^m P_0(t-x) dF(x)$$

$$(2.3) \quad P_k(t) = \int_0^t \sum_{j=0}^k \binom{m}{j} [H(t-x)]^{m-j} [1 - H(t-x)]^j P_{k-j}(t-x) dF(x)$$

therefore

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$$(2.4) \quad G(t, z) = \sum_{k=0}^{\infty} P_k(t) z^k$$

$$= [1 - F(t)] + \int_0^t \sum_{k=0}^{\infty} \sum_{j=0}^k \binom{m}{j} z^k [1 - H(t-x)]^j [H(t-x)]^{m-j} P_{k-j}(t-x) z^{k-j} dF(x) .$$

Interchanging summation yields

$$G(t, z) = [1 - F(t)] + \int_0^t \sum_{j=0}^m \sum_{k=j}^{\infty} \binom{m}{j} z^j [1 - H(t-x)]^j [H(t-x)]^{m-j} P_{k-j}(t-x) z^{k-j} dF(x) .$$

or

$$G(t, z) = [1 - F(t)] + \int_0^t G(t-x, z) [z + (1-z) H(t-x)]^m dF(x) \quad \text{q.e.d.}$$

If we differentiate $G(t, z)$ and evaluate at $z = 1$ then by exactly the same method as used in chapter 2 in the section summarizing Takacs' results we can prove:

Theorem 2: The binomial moments exist and can be obtained by the recurrence formula.

$$(2.5) \quad B_r(t) = \sum_{j=0}^r \binom{m}{j} \int_0^t B_{r-j}(t-x) [1 - H(t-x)]^j dF(x)$$

Cor. The mean of the ergodic distribution is simply $\frac{m}{\lambda} \rho$ where

$$\rho = \text{mean dead time} = \int_0^{\infty} x dH(x) < \infty \quad \text{and} \quad \lambda = \text{mean inter-arrival}$$

$$\text{time} = \int_0^{\infty} x dF(x) < \infty .$$

$$\text{Proof: As } B_0(t) = 1, \quad B_1(t) = \int_0^t \{B_1(t-x) + m[1 - H(t-x)]\} dF(x)$$

thus taking Laplace transforms yields

$$B_1(s) = B_r(s) \mathcal{Q}(s) + m \mathcal{Q}(s) L [1-H]$$

where $L[1-H]$ is the Laplace transform of $1 - H(x)$.

Hence $\beta_1(s) = m \frac{\mathcal{Q}(s)}{1-\mathcal{Q}(s)} L [1-H]$ and by inversion

$$B_1(t) = \int_0^t [1 - H(t-x)] dm(x)$$

where $m(x) = \sum_{n=0}^{\infty} F_n(x)$ and $F_n(x)$ denotes the n -fold convolution of $F(x)$.

Applying Smith's renewal theorem yields

$$(2.6) \quad B_1^* = \frac{m}{\lambda} \rho . \quad \text{q.e.d.}$$

The major problem in using theorem 2 for practical purposes is the presence of the j in the exponent of $[1 - H(t-x)]$ in formula (2.5). This difficulty will be high-lighted when we specialize to the case of exponential dead times in the next section.

§ 3. The Special Case of Exponential Dead Time and the Mean registration time for the type II counter with Max- m dead time.

In this section $H(x) = 1 - e^{-\mu x}$ so that (2.5) becomes

$$(3.1) \quad B_r(t) = \int_0^t \sum_{j=0}^r \binom{m}{j} B_{r-j}(t-x) e^{-\mu j(t-x)} dF(x) .$$

Upon taking Laplace transforms we obtain the recurrence equation for the Laplace transforms of the r -th binomial moments. Namely,

$$(3.2) \quad \beta_r(s) = \frac{\varphi(s)}{1-\varphi(s)} \sum_{j=1}^r \binom{m}{j} \beta_{r-j}(s+j\mu)$$

It is to be noted that this equation will in general be an m -th order difference equation with variable coefficients and will be difficult to solve unless $m = 1$.

We now proceed to discuss the imbedded Markov Chain in the case of exponential dead-time. We shall again see that the technical difficulties in this approach are due to the fact that the m will appear as an exponent in the integral equation determining the generating function of the queue size at arrivals.

If we let $\delta_n = \delta(\tau_n - 0)$ be the number of impulses present just before the n -th batch arrives the sequence $\{\delta_n\}$ forms a Markov Chain with transition probabilities

$$(3.3) \quad P(\delta_{n+r} = k | \delta_n = j) = P_{jk} = \int_0^{\infty} \binom{j+m}{k} e^{-\mu x k} (1-e^{-\mu x})^{j+m-k} dF(x) .$$

If we can find one solution $\{P_k\}$ to the system of equations

$$(3.4) \quad P_k = \sum_{j=0}^{\infty} P_j P_{jk}$$

such that $\{P_k\}$ is a probability distribution then by Foster's theorem [2] the chain will be ergodic and $\{P_k\}$ will be its actual unique limiting distribution, independent of initial conditions. Proceeding

as usual we introduce the generating functions $U(z) = \sum_{k=0}^{\infty} P_k z^k$. Since

$$P_k = \sum_{j=k-m}^{\infty} P_j \int_0^{\infty} \binom{j+m}{k} e^{-\mu x k} [1 - e^{-\mu x}]^{j+m-k} dF(x)$$

Multiplying by z^k and adding over k we obtain

$$U(z) = \sum_{j=0}^{\infty} \int_0^{\infty} \sum_{k=0}^{j+m} P_j \binom{j+m}{k} e^{-\mu x k} z^k (1 - e^{-\mu x})^{j+m-k} dF(x) \quad \text{or}$$

$$(3.5) \quad U(z) = \int_0^{\infty} (1 - e^{-\mu x} + z e^{-\mu x})^m U(1 - e^{-\mu x} + z e^{-\mu x}) dF(x)$$

Recalling that $B_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k$ is also equal to

$$\frac{1}{r!} \left. \frac{d^r U(z)}{dz^r} \right|_{z=1} \quad \text{and differentiating with respect to } z \quad r \text{ times in}$$

the given equation we conclude that the binomial moments of the limiting distribution satisfy the difference equation

$$(3.6) \quad B_r = \varrho(r\mu) \sum_{j=0}^r \binom{m}{r-j} B_j = \varrho(r\mu) \sum_{j=0}^r \binom{m}{j} B_{r-j}.$$

If we can solve (3.6) for the binomial moments then we can find the probabilities $\{P_k\}$ by Jordan's inversion formula. In particular P_0/μ is the mean recurrence time of the time between batches arriving and finding the counter free (or all servers free in the case of the

infinitely many server queue). However, this P_0/μ is also the mean time between successive registrations in the $\eta(t)$ process as $\eta(t) = 0$ iff. $\delta(t) = 0$. This is, of course, the reason we considered the batch arrival problem.

Notice that if we again set $\mu = 1$ $e_j = e(j\mu)$ $d_j = \frac{e(j\mu)}{1-e(j\mu)}$

(3.6) becomes

$$(3.7) \quad \beta_r = d_r \left\{ \sum_{j=1}^r \left(\begin{matrix} m \\ j \end{matrix} \right) B_{r-j} \right\} \quad \text{if } r \leq m$$

$$(3.7) \quad B_r = d_r \left\{ \sum_{j=1}^m \left(\begin{matrix} m \\ j \end{matrix} \right) B_{r-j} \right\} \quad \text{if } r \geq m$$

and $B_1 = m \frac{e_1}{1-e_1}$

$$B_2 = \frac{e_2}{1-e_2} \left[\frac{e_1}{1-e_1} m^2 + \left(\begin{matrix} m \\ 2 \end{matrix} \right) \right]$$

Theorem 3: The general solution to (3.7) is given by

$$(3.8) \quad B_r = \sum_{\text{all ordered partitions of } r \text{ into } r, r-1, \dots, 1 \text{ parts.}} d_{u_1} d_{u_1+u_2} \dots d_{u_1+u_2+\dots+u_k} \prod_{i=1}^k \left(\begin{matrix} m \\ u_i \end{matrix} \right)$$

i.e. for each $k \leq r$ consider partitions (u_1, \dots, u_k) . $\sum_{i=1}^k u_i = r$
 $1 \leq u_i \leq m$

proof: By definition $B_0 = 1$. $B_1 = md_1$ which is also given by our formula since there is only one partition of 1 namely $u_1 = 1$. For B_2 there are exactly two partitions of two (1,1) and (2) so our formula yields

$$\binom{m}{1}^2 d_1 d_2 + \binom{m}{2} d_2 \quad \text{agreeing with a "brute"}$$

force" computation. The general proof is by induction. Suppose the theorem is true for all $r \leq r-1$ we must show it is true for B_r .

$$\text{If } r \leq m \quad B_r = d_r \left\{ \sum_{j=1}^r \binom{m}{j} B_{r-j} \right\}$$

therefore

$$B_r = d_r \sum_{j=1}^r \binom{m}{j} \left\{ \begin{array}{l} \sum d_{u_1} d_{u_1+u_2} \cdots d_{u_1+\dots+u_k} \prod_{i=1}^k \binom{m}{u_i} \\ \text{of all ordered partitions} \\ \text{of } r-j \text{ into } k \text{ parts } k = 1, 2, \dots, r-j \\ \text{such that } \sum_{i=1}^k u_i = r-j \\ 1 \leq u_i \leq m \end{array} \right\}$$

$$= \sum \prod_{i=1}^k \binom{m}{u_i} d_{u_1} d_{u_1+u_2} \cdots d_{u_1+\dots+u_k}$$

all ordered partitions
of r into k parts $k = 1, \dots, r$
 $\sum_{i=1}^k u_i = r$

where the empty bracket is to be read as one. Notice that the $\binom{m}{j}$ term is just the last part of the partition of r into k parts i.e., $u_k = j$ and thus we see that the proof if $r \leq m$ is complete. The proof for $r \geq m$ is similar. As the formula (3.8) is true for $r = m$ by the previous proof we again use induction.

$$\begin{aligned}
 B_r &= d_r \left\{ \sum_{j=1}^m \binom{m}{j} B_{r-j} \right\} \quad \text{if } r \geq m \\
 &= d_r \sum_{j=1}^m \binom{m}{j} \sum_{\substack{\text{all ordered partitions} \\ \text{of } r-j \text{ into } k \text{ parts} \\ k = 1, \dots, r-j \text{ such that} \\ \sum_{i=1}^k u_i = r-j \quad 1 \leq u_i \leq m}} \frac{1}{i!} \binom{m}{u_1} d_{u_1} d_{u_1+u_2} \dots d_{u_1+\dots+u_k}
 \end{aligned}$$

since the $\binom{m}{j}$ term again becomes the last part j of the total partition of r and combines with all partitions of $r - j$ to give a partition of r into $k + 1$ parts. Note that for some k , especially small ones, there may be no relevant partitions as the partition of r may have u_i 's $> m$ and $\binom{m}{u_1} = 0$ if $u_1 > m$. Actually it is not surprising that (3.7) and (3.7') have the same solution as they are identical if we make the convention that $B_{-k} = 0$ if $k > u$.

We now are in position to give the formula for the mean of $R(x)$.

Theorem 4: The mean time between successive registrations in the type II Particle Counter where the arrivals form a recurrent process i.e.

$P(\tau_n - \tau_{n-1} \leq x) = F(x)$ $\lambda = \int_0^\infty x dF(x) < \infty$ and the impulse times are max-m distributed is

$$\frac{\lambda}{P_0} \quad \text{where}$$

$$P_0 = \sum_{r=0}^{\infty} (-1)^r B_r$$

where the B_r are given by (3.8) .

Proof: By the fundamental theorem of recurrent events applied to Markov Chains the mean number of steps in the chain before a return to E_0 starting from E_0 is $1/P_0$. However the length of time between successive steps of the chain is distributed as $F(x)$. If we regard the $R(x)$ distribution to be the sum of $N F(x)$'s where N = the number of steps in the chain (or number particles that arrive) until an arriving particle finds the counter free we can apply Wald's Fundamental Identity to conclude that

$$E(R) = \frac{\lambda}{P_0} \quad \text{as} \quad \lambda = \int_0^\infty x dF(x) < \infty .$$

Notice that the event $\{N = n\}$ depends only on the first $n-1$ $F(x)$ variables and thus the use of Wald's identity is justified. q.e.d

$$(3.9) \quad \text{Since } P_0 = \sum_{r=0}^{\infty} (-1)^r B_r$$

and as $B_r \leq \frac{C^r}{r!}$ for some constant C the series for P_0 converges.

The usefulness of our approach depends on the rapidity of the convergence, not just because we want to add up as few terms as possible but also because the B_r 's are difficult and tedious to compute for large r . Nevertheless, by the introduction of an auxiliary stochastic process, we have been able to give a precise formula λ/P_0 for the mean time between successive registrations.

One standard device used in particle counting is a scaling circuit which lets only every r -th particle through to the counter. This, mathematically speaking, transforms the usual Poisson input into Erlang- r input. In this case

$$d(s) = \frac{e(s)}{1-e(s)} = \frac{(\lambda/\lambda+s)^r}{1-(\lambda/\lambda+s)^r}$$

and $B_0=1$.

$$B_1 = \frac{m(\lambda/\lambda+1)^r}{1-(\lambda/\lambda+1)^r}$$

$$B_2 = m^2 \frac{(\lambda/\lambda+1)^r}{1-(\lambda/\lambda+1)^r} \cdot \frac{(\lambda/\lambda+2)^r}{1-(\lambda/\lambda+2)^r} + \left(\frac{m}{2}\right) \frac{(\lambda/\lambda+2)^r}{1-(\lambda/\lambda+2)^r}$$

If $\lambda = \mu = 1$ $m = 2$ $r \sim 64$ then $B_2 < \frac{1}{200}$.

The usefulness of (3.9) therefore depends on the relative sizes of m and r . In order to approximate a true dead time distribution a large m will be needed as

the mean of a max-m distribution is $\sum_{j=1}^m \frac{1}{j}$

the variance " " " $\sum_{j=1}^m 1/j^2$

and most dead time distributions will be "almost constant" and thus the mean must be much larger than the variance.

§ 4. Using the $\eta(t)$ Process to Investigate the $\delta(t)$ process when arrivals are Poisson

So far we have simply used the "batch arrivals" process as a tool to determine the mean time between consecutive registrations for the type II counter with general input and max-m impulse time. The present section is devoted to a minor reversal of this procedure.

Suppose we really were interested in the infinitely many server queueing process where arrivals obey a Poisson law with parameter λ , service time is exponential with parameter μ and the arrivals are in batches of m . The standard approach [3] yields the following set of difference equations for the limiting distribution of the queue size.

$$\text{if } k \leq m - 1 \quad (k + 1)\mu P_{k+1} = (\lambda + k\mu)P_k$$

$$\text{if } k \geq m \quad (k + 1)\mu P_{k+1} = (\lambda + k\mu) - P_{k-m}$$

The usual method is to solve these equations in terms of P_0 and then normalize using the condition $\sum_{i=0}^{\infty} P_i = 1$. In our case

it is difficult to obtain the general solution but P_0 is known to be equal to $e^{-\lambda \rho}$ where ρ = mean of the max-m distribution. The justification for this assertion is that if we consider the two processes in operation simultaneously we are in 0 in the δ process if and only if we are in state 0 of the η process $G/\max m/\infty$. By Takács' result [1] $P_0 = e^{-\lambda \rho}$. In fact, Takács shows that the limiting queue size for an infinitely many server queue with Poisson arrivals and general service distribution $H(x)$ is Poisson with parameter $\rho = \int_0^\infty x dH(x) < \infty$.

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Chapter V: Attempts to Determine the Variance of the
"Time between Consecutive Registrations."

§ 1. Introduction:

In order to obtain the asymptotic distribution of \mathcal{V}_t , the number of registered particles at time t , we need not only the mean of $R(x)$ but also the variance of $R(x)$. Although we have not been able to obtain a formula for the variance of $R(x)$, we can give an approximation to the variance for Erlang- r inter-arrival distributions if r is large.

Despite the fact that we were unsuccessful in our attempt to discover the variance of $R(x)$, it is informative to review the approaches used and to see the difficulties involved. The first method uses Wald's Fundamental Identity. Let us consider the δ process. When the first batch of arrivals comes we keep on sampling from the inter-arrival distribution until an arriving batch finds the counter free. If N = the number of arrivals between the registrations including the 2nd registered particle [i.e. $N = n$ means that after registration 1, $n-1$ particles were not counted but the n -th was], then N = the recurrence time of the state E_0 of the imbedded chain $\{\delta_n\}$. If θ stands for the inter-arrival distribution and $S_N = \theta_1 + \theta_2 + \dots + \theta_N$ then S_N has the distribution $R(x)$. We used this idea previously in showing that

$$E(R) = \lambda/P_0 \text{ as } E(\theta) = \lambda \text{ and } E(N) = 1/P_0.$$

Assuming that both N and θ have finite second moments we can differ-

entiate Wald's Identity [see Harris [1]] and obtain the following formula for $E[S_N^2]$:

$$E[S_N^2] = 2E(NS_N)E(\theta) - E[N^2]E^2(\theta) + E(N)V(\theta)$$

The main difficulty encountered in this approach is the determination of $E(NS_N)$. The quantity $E(N^2)$ is also hard to compute; nevertheless, we can give a formula for $\sigma^2(N)$ and thus for $E(N^2)$. Because $\sigma^2(N)$ is very complicated this method will not be of practical use even if we could solve for $E[NS_N]$. It is of interest to realize that the random variable N can be defined to be the number of steps between successive transitions $E_0 \rightarrow E_m$ in the chain $\{\delta_n\}$.

In the second part of the chapter we outline an approach using integral equations, also developed by Takács. Unfortunately, this second approach yields less than the first.

§ 2. An Approach via Wald's Fundamental Identity and Recurrent Events.

As already indicated, the present approach is an attempt to compute the second moment of $R(x)$ by use of the formula

$$(2.1) E[R^2] = E[S_N^2] = 2E[NS_N]E(\theta) - E[N^2]E^2(\theta) + E(N)V(\theta)$$

We shall first determine $\sigma^2(N) = E(N^2) - E^2(N)$, the variance of the number of steps between consecutive transitions $E_0 \rightarrow E_m$ in the

imbedded chain $\{\delta_n\}$.

Recall that $p_{jk} = \int_0^\infty \binom{j+m}{k} e^{-\mu k x} (1 - e^{-\mu x})^{j+m-k} dF(x)$.

Let $\{P_k^{(n)}\}$ be the distribution of the number of impulses present in the δ process just before the n -th arrival. Starting from the initial distribution $\{P_k^{(1)}\}$ the distributions $\{P_k^{(n)}\}$ can be determined recursively by the formulas.

$$(2.1) \quad P_k^{(n+1)} = \sum_{j=k-\infty}^{\infty} P_{jk} P_j^{(n)} \quad n = 1, 2, 3, \dots$$

Consider the binomial moments of the distribution of the queue size at the n -th step

$$(2.2) \quad B_r^{(n)} = E \left\{ \binom{\delta_n}{r} \right\} = \sum_{k=r}^{\infty} \binom{k}{r} P_k^{(n)}.$$

Using the elementary result that the binomial moments of the Binomial (Bernoulli) distribution $Q_k = \binom{n}{k} p^k (1-p)^{n-k}$ are just $B_r = \binom{n}{r} p^r$ we can prove:

Theorem: $B_0^{(n)} = 1 \quad n = 1, 2, \dots$ and

$$(2.3) \quad B_r^{(n+1)} = \varphi_r \sum_{k=0}^m \binom{m}{k} B_{r-k}^{(n)} \quad \text{where } B_{-\nu}^{(n)} = 0 \text{ if } \nu > 0.$$

Proof: If we let $\theta_n = \tau_n - \tau_{n-1}$, as usual, then conditional upon

$$\theta_n = x \quad \text{and} \quad \delta_n = j$$

$$E[(\delta_{n+1}^j) | \delta_n = j, \theta_n = x] = (j+m) e^{-r\mu x}$$

because under the given conditions δ_{n+1} is just a Binomial variable with parameters $j+m$ and $e^{-\mu x}$ thus

$$E[(\delta_{n+1}^j) | \delta_n = j] = (j+m) \int_0^\infty e^{-r\mu x} dF(x) = (j+m) \varphi_r.$$

$$B_r^{(n+1)} = \sum_{j=0}^{\infty} P_j^{(n)} (j+m) \varphi_r = \varphi_r \sum_{j=0}^{\infty} (j+m) P_j^{(n)}.$$

$$\text{As } (j+m) = \sum_{k=0}^m \binom{j}{r-k} \binom{m}{k}$$

$$B_r^{(n+1)} = \varphi_r \sum_{k=0}^m \binom{m}{k} \left\{ \sum_{j=0}^{\infty} \binom{j}{r-k} P_j^{(n)} \right\}$$

$$B_r^{(n+1)} = \varphi_r \sum_{k=0}^m \binom{m}{k} B_{r-k}^{(n)} \quad \text{q.e.d.}$$

If $n \rightarrow \infty$ and $B_r^* = \lim_{n \rightarrow \infty} B_r^{(n)}$ then the B_r^* satisfy equations previously derived for the ergodic distribution (see Chapter IV equation (3.6)). Starting from $B_r^{(1)}$ $r = 1, 2, \dots$ the binomial moments of the number of impulses present at the first arrival we can use (2.3) to obtain the binomial moments of the number of impulses present just before the n -th arrival (remember we are working with the $\delta(t)$ process). If $\delta(0) = 1$ and $\tau_1 = x$, δ_1 has a Bernoulli

distribution with parameters 1 and $e^{-\mu x}$ and thus:

$$(2.4) \quad B_r^{(1)} = \left\{ \binom{\delta_1}{r} \right\} = \binom{1}{r} \varphi_r \quad r = 0, 1, 2, \dots$$

We proceed to determine the generating function for the r -th binomial moment of $\{\delta_n\}$. Let $B_r(w) = \sum_{n=1}^{\infty} B_r^{(n)} w^n$.

Lemma: Suppose $\delta(0) = 1$ then the $B_r(w)$ satisfy the difference equation:

$$(2.5) \quad B_r(w) = \frac{w \varphi_r}{1 - w \varphi_r} \left\{ \binom{1}{r} \varphi_r + \sum_{k=1}^m \binom{m}{k} B_{r-k}(w) \right\}.$$

Proof:

$$B_r^{(n+1)} = \varphi_r \sum_{k=0}^m \binom{m}{k} B_{r-k}^{(n)}.$$

Multiplying by $w^{(n+1)}$ and summing over n we obtain

$$\sum_{n=1}^{\infty} B_r^{(n+1)} w^{n+1} = \sum_{n=1}^{\infty} \varphi_r w \sum_{k=0}^m \binom{m}{k} B_{r-k}^{(n)} w^n$$

or

$$B_r(w) - \binom{1}{r} \varphi_r w = \varphi_r w \sum_{k=0}^m \binom{m}{k} B_{r-k}(w).$$

hence

$$B_r(w) = \frac{w \varphi_r}{1 - w \varphi_r} \left[\binom{1}{r} + \sum_{k=1}^m \binom{m}{k} B_{r-k}(w) \right] \quad \text{q.e.d.}$$

For the practical purpose of particle counting theory we

we may assume that initially we are in state E_0 i.e. $\delta(0) = 0$.

Thus the equations to be solved are

$$(2.6) \quad B_r(w) = d_r(w) \left\{ \sum_{k=1}^m \binom{m}{k} B_{r-k}(w) \right\}$$

$$B_0(w) = \frac{w}{1-w}$$

where $d_r(w) = \frac{\varphi_r^w}{1-\varphi_r^w}$ and $d_0(w) = \frac{w}{1-w}$.

Theorem: The solutions of the equations (2.6) for the determination of the generating functions of the transient behavior of the binomial moments of $\{\delta_n\}$ are given by:

$$(2.7) \quad B_r(w) = \sum \prod_{i=1}^k \binom{m}{u_i} d_0(w) d_{u_1}(w) d_{u_1+u_2}(w) \dots d_{u_1+\dots+u_k}(w)$$

all ordered partitions (u_1, \dots, u_k)
of r into k parts, $k = 1, \dots, r$
such that $\sum_{i=1}^k u_i = r$ and $1 \leq u_i \leq m$

The proof is omitted as it is exactly the same as the one given for the imbedded chain except that d_r is now replaced by $d_r(w)$. This accounts for the presence of $d_0(w) = \frac{w}{1-w}$ in the formula (2.7).

As $B_0 = 1$, $d_0 = 1$ and so d_0 failed to appear in (3.8) of chapter IV.

Since $B_r(w)$ is analytic in a neighborhood of the origin,

$$P_0(w) = \sum_{n=1}^{\infty} P_0^{(n)} w^n = \sum_{r=0}^{\infty} (-1)^r B_r(w) \quad \text{will also be analytic}$$

in a neighborhood of the origin. In theory, therefore, we can use Cauchy's integral formula for derivatives to determine the

$\{P_0^{(n)}\}$ $n = 1, 2, \dots$. Let us again consider that every step in the chain is one trial in a recurrent event scheme where the recurrent event \mathcal{E} is "we are in state 0 at the n -th trial" (or step of the chain). The random variable N which equals the number of trials between successive occurrences of \mathcal{E} and which equals the number of steps between consecutive transitions $E_0 \rightarrow E_m$ in the chain $\{\delta_n\}$ is, of course, the object of our discussion. We already know that $E(N) = 1/P_0$. By a problem in Feller's text [2] we can also determine the variance of N . Specifically

$$(2.8) \quad \sum_n (P_0^{(n)} - P_0) = \frac{\sigma^2(N) - 1/P_0 + (1/P_0)^2}{2(1/P_0)^2}$$

where P_0 is the P_0 of the ergodic distribution and is given in section 3 of Chapter IV. Another expression for the right side of (2.8) is

$$(2.9) \quad \frac{\sigma^2(N) - 1/P_0 + (1/P_0)^2}{2(1/P_0)^2} = \lim_{w \rightarrow 1} \sum_{n=1}^{\infty} (P_0^{(n)} - P_0) w^n$$

$$= \lim_{w \rightarrow 1} \left\{ \sum_{r=0}^{\infty} (-1)^r B_r(w) - \left(\sum_{r=0}^{\infty} (-1)^r B_r \right) \frac{w}{1-w} \right\}$$

Unfortunately, the determination of $\sigma^2(N)$ in this manner requires far more computation than the determination of P_0 because $\sigma^2(N)$ depends on the sum $\sum_{n=1}^{\infty} (P_0^{(n)} - P_0)$. Also even if $P_0^{(n)}$ is "close to" P_0 many terms will be needed to assure that

$\sum_{n=1}^S (P_0^{(n)} - P_0)$ is close to $\sum_{n=1}^{\infty} (P_0^{(n)} - P_0)$. In theory, never-

theless, we have determined both the mean and variance of the recurrence time of the 0-state in the imbedded Markov Chain of the $\delta(t)$ process. Although we were able to deduce that the mean time between successive registrations in both the $\delta(t)$ and the $\eta(t)$ processes is equal to $1/P_0$, we have been unable to discover an expression for the variance of the "time between successive registrations." The difficulty arises from the fact that Wald's Fundamental Identity of Sequential Analysis yields an intractable expression for the second moment of the sum of a random number of random variables when the number chosen depends on a sequential stopping rule. Under suitable conditions (see Harris [1]) we have

$$(2.10) \quad E[S_N^2] = 2E(NS_N)E(\theta) - E(N^2)E^2(\theta) + E(N)V(\theta)$$

where N = the number of random variables θ_n , all of which are independent and identically distributed as $F(x)$, chosen. The term $E(NS_N)$ is well known to be the "troublemaker."

In an important special case we can, however, approximate $E(NS_N)$.

Suppose we have a type II counter with max-m impulse time distribution and we are counting particles arriving according to a Poisson Process. If the particles pass through a scalar which lets only every r-th particle through then the input to the counter is an Erlang-r process. If r is large it will almost be a constant. Thus we now calculate $E(NS_N)$ if the inter-arrival distribution is a constant.

Lemma: If the input process is a constant i.e. the time between consecutive arrivals is $\lambda > 0$. Then

$$(2.11) \quad E[NS_N] = E(N^2)\lambda.$$

Proof: Under the hypotheses of the lemma $P_r[N=n] = P_r[N=n, S_N=n\lambda]$ thus

$$\begin{aligned} E(NS_N) &= \sum_{n=1}^{\infty} n(\lambda n) P_r(N=n, S_N=n\lambda) \\ &= \sum_{n=1}^{\infty} n^2 \lambda P_r(N=n) = \lambda E(N^2) \end{aligned}$$

When the arrivals have an Erlang-r inter-arrival distribution, therefore, we suggest approximating $E(NS_N)$ by $\lambda E(N^2)$ in equation (2.10). As $E(\theta) = r/\lambda = \lambda$ and $V(\theta) = r/\lambda^2$ as θ is now Erlang-r we obtain from (2.10)

$$\begin{aligned} \text{Var}(S_N) &= E[S_N^2] - E^2[S_N] = 2\lambda E(N^2)E(2) - E(N^2)E^2(\theta) \\ &\quad + E(N)V(2) - E^2(S_N) \\ &= \frac{r^2}{\lambda^2} E(N^2) + E(N^2) + E(N^2) + E(N) \frac{r}{\lambda^2} - (\lambda/P_0)^2 \end{aligned}$$

Thus if $V(\theta) = r/\lambda^2$ is small (thus how large r must be for this approximation to be used depends on λ) we can say that approximately $\sigma^2(R)$, the variance of the time between consecutive registrations is:

$$(2.12) \quad \sigma^2(R) = \frac{r^2}{\lambda^2} \sigma^2(N) + \frac{1}{p_0} \frac{r}{\lambda^2}$$

Since the times $\{\tau_n\}$ of registrations of a particle form a recurrent process, with $P\{\tau_{n+1} - \tau_n \leq x\} = R(x)$ and if we denote by ν_t , the number of registrations by time t then Feller's Central Limit theorem for recurrent events (theorem 3 of chapter I) asserts that if $\sigma^2(R) < \infty$ then the distribution of ν_t is asymptotically normal i.e.

$$(2.13) \quad \lim_{t \rightarrow \infty} P\left\{ \frac{\nu_t - t/\mu}{\sqrt{\frac{\sigma^2(R)}{\mu^2} t}} \leq x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du$$

where $\mu = E(R)$ $\sigma^2 = \sigma^2(R)$.

Applying this result to the type II counter with Erlang- r inter-arrival times and max- m dead time distributions the number ν_t of registrations by time t has approximately the asymptotic normal distribution with parameters,

$$\mu = \mu(R) = \frac{r}{\lambda} \cdot \frac{1}{p_0}$$

$$\sigma^2 = \sigma^2(R) = \frac{r^2}{\lambda^2} \sigma^2(N) + \frac{1}{p_0} \frac{r}{\lambda^2}$$

In the paragraph above we have blithely written $1/P_0$ and $\sigma^2(N)$. It must be remembered that even in the case of Erlang-r input they are not at all easy to compute.

§ 2. The registration time Distribution in the Case of Constant Inter-arrival times.

So far we have devoted our attention to the case where $F(x)$ is non-lattice. In considering the variance of the time between successive registrations for Erlang-r input, we used constant arrivals to approximate Erlang-r input and were able to obtain an approximation for $\sigma^2(R)$ and thus give an approximation for the asymptotic distribution of the number of registrations. When the inter-arrival distributions is constant (α) it is easy to derive the exact distribution of the time R between successive registrations. We shall again work with the δ process, the arriving particle produces m independent exponentially distributed (parameter μ) impulses.

Theorem:
$$P[R = k\alpha] = e^{-k\mu\alpha} \sum_{\substack{\text{all } (j_1, \dots, j_k) \\ \text{such that } j_1 < m \quad j_1 + j_2 < 2m \\ \dots \quad j_1 + \dots + j_{k-1} < (k-1)m \\ \text{and } j_1 + \dots + j_k = km}} \frac{(\mu\alpha)^{km}}{j_1! j_2! \dots j_k!}$$

Proof: In order that the k -th arriving particle after a registration be the first one registered there must be at least one impulse present at the times when particles $1, 2, \dots, k-1$ arrive. Therefore, at time α no more than $(m-1)$ of the original impulses can have ended i.e. j_1 impulses can expire by time α where $j_1 = 0, 1, \dots, m-1$ but not m . In the next time interval $(\alpha, 2\alpha]$ j_2 impulses end but if the second arrival is not to be registered $j_2 < m - j_1 + m$ or $j_1 + j_2 < 2m$. Similarly we see that in the i -th ($i < k$) interval of length α , $[(i-1)\alpha, i\alpha]$, j_i impulses and j_i satisfies $j_1 + j_2 + \dots + j_i < im$. Finally if the k -th particle is the first to be registered, j_k = the number of impulses expiring in $[(k-1)\alpha, k\alpha]$ is equal to $j_k = km - (j_1 + \dots + j_{k-1})$. Therefore:

$$P[R = k\alpha] = \sum_{\substack{\text{all } (j_1, \dots, j_k) \text{ such that} \\ j_1 < m, j_1 + j_2 < 2m, \dots, j_1 + \dots + j_{k-1} < (k-1)m \\ j_1 + \dots + j_k = km}} \prod_{i=1}^k e^{-\mu\alpha} \frac{(\mu\alpha)^{j_i}}{j_i!}$$

$$= e^{-k\mu\alpha} (\mu\alpha)^{km} \sum_{\substack{\text{all } (j_1, \dots, j_k) \text{ such that} \\ j_1 < m, \dots, j_1 + \dots + j_{k-1} < (k-1)m \\ j_1 + \dots + j_k = km}} \prod_{i=1}^k \frac{1}{j_i!} \quad \text{q.e.d.}$$

To the best of the author's knowledge there is no known closed form for this sum.

References

- [1] Harris, T.E. (1947) "Note on differentiation under the expectation sign in the fundamental identity of Sequential Analysis.
Ann. Math. Statist. 18 pp 294-295.

- [2] Feller, William (1957) An Introduction to Probability Theory and its Applications 1 (2nd ed.) Wiley, New York.

Chapter 5B: The Integral Equation Approach

Although we have been able to obtain the mean of the "time between successive registrations" assuming that the impulse time obeyed a max-m law, we could not obtain the variance since it depends on $E[NS_N]$. In 1957, Professor Takács suggested a new approach which leads to integral equations for the mean and variance of the times between consecutive registrations; conditional on the impulse time produced by the previous registrant. After summarizing Takács' methods we solve the integral equations for a max-2 distribution. The situation, even for max-2 dead time is surprisingly complicated.

§ 2.

A Summary of Takács' Results

We assume, as usual, that particles arrive at a counter at times $\tau_0, \tau_1, \tau_2, \dots$; each one produces an impulse of duration $\{\lambda_n\}$ where $P\{\lambda_n \leq x\} = H(x)$ and the $\{\lambda_n\}$ are identically distributed and independent of one another and of the $\{\tau_n\}$ sequence. We denote the subsequence of $\{\tau_n\}$ $n = 0, 1, 2, \dots$ of those particles actually registered by $\{\tau'_n\}$ $n = 0, 1, \dots$. When the times $\{\tau_n - \tau_{n-1}\}$ $n = 1, 2, \dots$ between successive arrivals are identically distributed as $F(x)$ then the time differences $\{\tau'_n - \tau'_{n-1}\}$ between successive registrations will also be identically distributed random variables say $R(x)$. If we let

$$\mu = \int_0^{\infty} x dF(x) \qquad \sigma^2 = \int_0^{\infty} (x-\mu)^2 dF(x)$$

then our objective is to find

$$(2.1) \quad A = M[\tau'_n - \tau'_{n-1}] = \int_0^{\infty} x dR(x)$$

$$(2.2) \quad B^2 = D^2[\tau'_n - \tau'_{n-1}] = \int_0^{\infty} (x-A)^2 dR(x)$$

Once we've found A and B^2 we can apply Feller's Central Limit theorem of renewal theory to derive the asymptotic distribution of N_t = the number of registrations occurring in the time interval $(0, t]$, provided of course, that A and B^2 are finite. To determine A and B^2 we introduce the following conditional expectations.

$$(2.3) \quad A(y) = M[\tau'_1 | \chi_0 = y]$$

$$(2.4) \quad B^2(y) = D^2[\tau'_1 | \chi_0 = y]$$

Knowing these, by the theorem of total expectation we have

$$(2.5) \quad A = \int_0^{\infty} A(y) dH(y)$$

and

$$(2.6) \quad B^2 = \int_0^{\infty} B^2(y) dH(y) + \int_0^{\infty} [A(y) - A]^2 dH(y)$$

As the length of time between time 0 (when the event τ_0 occurs) and the time that the next particle comes and produces an impulse is distributed as $F(x)$ we have

$$(2.7) \quad C(y) = M[\tau_1 | \chi_0 = y] = \mu$$

$$(2.8) \quad D^2(y) = D^2[\tau_1 | \chi_0 = y] = \sigma^2$$

Theorem: The conditional expectation $A(y) = M[\tau_1' | \lambda_0 = y]$ can be determined with the aid of the following integral equation:

$$(2.9) \quad A(y) = \int_0^y A(y-x)H(y-x)dF(x) + \int_0^y \left[\int_{y-x}^{\infty} A(z)dH(z) \right] dF(x) + C(y)$$

Proof:

We have

$$M[\tau_1' | \tau_1 - \tau_0 = x, \lambda_0 = y, \lambda_1 = z] = \begin{cases} x + A(y-x) & \text{if } 0 < z \leq y-x \text{ and } 0 < x \leq y \\ x + A(z) & \text{if } y-x \leq z < \infty \text{ and } 0 < x \leq y \\ x & \text{if } y < x < \infty \end{cases}$$

and (2.9) follows by the theorem of total expectation.

Theorem: The conditional variance $B^2(y) = D^2[\tau_1' | \lambda_0 = y]$ can be determined from the integral equation;

$$(2.10) \quad B^2(y) = \int_0^y B^2(y-x)H(y-x)dF(x) + \int_0^y \left[\int_{y-x}^{\infty} B^2(z)dH(z) \right] dF(x) \\ + \int_0^y [x + A(y-x)]^2 H(y-x) dF(x) + \int_0^y \left[\int_{y-x}^{\infty} [x + A(z)]^2 dH(z) \right] dF(x) + D^2(y) .$$

Proof: We have

$$D^2[\tau_1' | \tau_1 - \tau_0 = x, \lambda_0 = y, \lambda_1 = z] = \begin{cases} B^2(y-x) & \text{if } 0 < z \leq y-x \text{ and } 0 < x \leq y \\ B^2(z) & \text{if } y-x \leq z < \infty \text{ and } 0 < x \leq y \\ 0 & \text{if } y < x < \infty \end{cases}$$

and the result again follows by the theorem of total expectation.

It should be recognized that equations (2.9) and (2.10) are of the same type, since once we've solved (2.9) we can let

$$r(y) = \int_0^y [x + A(y-x)]^2 H(y-x) dF(x) + \int_0^y \left\{ \int_{y-x}^{\infty} [x+A(z)]^2 dH(z) \right\} dF(x) + D^2(y)$$

for this will simply be a function of y .

Then

$$B^2(y) = \int_0^y B^2(y-x) H(y-x) dF(x) + \int_0^y \left[\int_{y-x}^{\infty} B^2(z) dH(z) \right] dF(x) + r(y)$$

which is identical in form with (2.9).

§ 3. The application of the method to special dead time distributions.

In this section we shall treat the case of exponentially distributed dead time in full detail. Then we proceed to formulate the problem for max-m impulse times and solve the resulting equations if $m=2$. Finally we show how the Erlang-2 case leads to a still more complicated equation.

Example: $H(x) = 1 - e^{-\mu x}$

We let $\frac{\psi(s)}{s} = \int_0^{\infty} e^{-sy} A(y) dy$ $r(s) = \int_0^{\infty} e^{-sy} dC(y)$

and $\Omega(s) = \int_0^{\infty} e^{-sy} dF(y)$.

Equation (2.9) becomes

$$(3.1) \quad A(y) = \int_0^y A(y-x)H(y-x)dF(x) + C(y) + \int_0^y \int_{y-x}^{\infty} A(z)dH(z)dF(x)$$

Taking Laplace transforms we obtain

$$\frac{\Psi(s)}{s} = \frac{\Omega(s)\Psi(s)}{s} - \frac{\Omega(s)}{s} \frac{\Psi(s+\mu)}{s+\mu} + \frac{\Gamma(s)}{s} + \frac{\Omega(s)}{s} [\Psi(\mu) - \frac{\Omega(\mu)}{s+\mu}]$$

or

$$(3.2) \quad \Psi(s) = -\Omega(s)\Psi(s+\mu) + \Omega(s)\Psi'(s) + \Psi(\mu)\Omega(s) + \Gamma(s)$$

and finally

$$(3.3) \quad \Psi(s+\mu) = \Psi(\mu) + \frac{\Gamma(s)}{\Omega(s)} - \frac{1-\Omega(s)}{\Omega(s)} \Psi(s)$$

By successive applications of this formula we can express $\Psi(s+n\mu)$ in terms of $\Psi(s)$ and $\Psi(\mu)$. This leads to

$$(3.4) \quad \Psi(s+(n+1)\mu) = \Psi(\mu) + \frac{\Gamma(s+n\mu)}{\Omega(s+n\mu)} - \left[\frac{1-\Omega(s+n\mu)}{\Omega(s+n\mu)} \right] \Psi(s+n\mu)$$

$$\text{If we let } d_n = \Psi(\mu) + \frac{\Gamma(s+n\mu)}{\Omega(s+n\mu)}; \quad \omega_n = \frac{1-\Omega(s+n\mu)}{\Omega(s+n\mu)}$$

this simplifies to the recursion relation:

$$(3.5) \quad \Psi(s + (n+1)\mu) = d_n - \omega_n \Psi(s+n\mu)$$

Theorem: The general solution to (3.5) is:

$$(3.6) \quad \psi(s+(n+1)\mu) = d_n + (-1)^{n+1} \omega_n \omega_{n-1} \cdots \omega_0 \psi(s) + \sum_{i=0}^{n-1} \omega_n \cdots \omega_{i+1} d_i (-1)^{n+1-i}$$

Proof: If $n = 1$, the formula (3.6) satisfies (3.5). Assume $\psi(s+k\mu)$ is given by (3.6) for $k = 1, \dots, n$. We now show $\psi(s+(n+1)\mu)$ is also given by (3.6). By the induction hypothesis

$$\psi(s+\mu) = d_{n-1} + (-1)^n \omega_{n-1} \cdots \omega_0 \psi(s) + \sum_{i=0}^{n-2} \omega_{n-1} \cdots \omega_{i+1} d_i (-1)^{n-1-i}$$

Substituting in (3.5) we obtain

$$\begin{aligned} \psi(s+(n+1)\mu) &= d_{n+1} + \omega_n d_{n-1} + (-1)^{n+1} \omega_n \cdots \omega_0 \psi(s) + \sum_{i=0}^{n-2} \omega_n \omega_{n-1} \cdots \omega_{i+1} d_i (-1)^{n+1-i} \\ &= d_{n+1} + (-1)^{n+1} \omega_n \cdots \omega_0 \psi(s) + \sum_{i=0}^{n-1} \omega_n \cdots \omega_{i+1} d_i (-1)^{n+1-i} \end{aligned}$$

as desired.

As $\psi(s)$ is a Laplace transform

$$\lim_{n \rightarrow \infty} \psi(s+n\mu) = 0. \quad \text{Similarly} \quad \lim_{n \rightarrow \infty} \Omega(s+n\mu) = 0$$

Thus letting $n \rightarrow \infty$ in (3.6) we conclude that

$$\psi(s) = \sum_{i=0}^{\infty} (-1)^i d_i \omega_i \omega_{i-1} \cdots \omega_0$$

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or

$$(3.7) \quad \psi(s) = \sum_{i=0}^{\infty} (-1)^i \left[\psi(\mu) + \frac{\Gamma(s+i\mu)}{\Omega(s+i\mu)} \right] \left(\prod_{j=0}^i \frac{\Omega(s+j\mu)}{1-\Omega(s+j\mu)} \right)$$

If we set $s = \mu$ we find that

$$(3.8) \quad \psi(\mu) = \frac{\sum_{i=1}^{\infty} (-1)^{i-1} \frac{\Gamma(i\mu)}{\Omega(i\mu)} \prod_{j=1}^i \frac{\Omega(j\mu)}{1-\Omega(j\mu)}}{1 - \sum_{i=1}^{\infty} (-1)^{i-1} \prod_{j=1}^i \frac{\Omega(j\mu)}{1-\Omega(j\mu)}}$$

Thus $A(y)$ can be determined by inversion of (3.7). It is of interest to note that

$$(3.9) \quad A = \int_0^{\infty} A(y) \mu e^{-\mu y} dy = \psi(\mu) .$$

The conditional variance $B^2(y)$ may likewise be determined by Laplace-Stieltjes transforms.

Example 2:
$$H(x) = \sum_{j=0}^m (-1)^j \binom{m}{j} e^{-\mu j x}$$

The equation to be solved is

$$(3.10) \quad A(y) = \int_0^y A(y-x) H(y-x) dF(x) + C(y) + \int_0^y \int_{y-x}^{\infty} A(z) dH(z) dF(x)$$

Forming Laplace-Stieltjes transforms we obtain:

$$(3.11) \quad \psi(s) \frac{[1-\Omega(s)]}{\Omega(s)} = \frac{\Gamma(s)}{\Omega(s)} + \sum_{j=1}^m (-1)^j \binom{m}{j} [\psi(s+j\mu) - \psi(j\mu)]$$

If $m = 2$ the above equation becomes

$$\psi(s+2\mu) = 2\psi(s+\mu) + \psi(s)\omega_0 + \psi(2\mu) - 2\psi(\mu) + \gamma_0$$

$$\text{where} \quad \omega_k = \frac{1-\Omega(s+k\mu)}{\Omega(s+k\mu)} \quad \gamma_k = \frac{\Gamma(s+k\mu)}{\Omega(s+k\mu)}$$

Theorem:

$$(3.12) \quad \psi(s+n\mu) = \psi(s+\mu) \left[\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-1-2k} \left\{ \begin{array}{l} \text{sum of products of exactly } k \\ \text{terms from } \omega_1, \dots, \omega_{n-2} \text{ chosen} \\ \text{such that no two adjacent} \\ \omega_1 \text{'s are taken} \end{array} \right\} \right. \\ + \psi(s) \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \left\{ \begin{array}{l} \text{sums of products of exactly } k \\ \text{terms from } \omega_1, \dots, \omega_{n-2} \text{ chosen} \\ \text{such that } \omega_0 \text{ is always present} \end{array} \right\} \\ + [\psi(2\mu) - 2\psi(\mu)] \left[(2^{n-1}-1) + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left\{ \begin{array}{l} k\text{-tuples from } \omega_2, \dots, \omega_{n-2} \text{ such} \\ \text{that the coefficient of the } k\text{-tuple} \\ \text{beginning with } \omega_r, r \geq 2 \text{ is} \\ 2^{n-2k-1} [1 - (\frac{1}{2})^{r-1}] \end{array} \right\} \right] \\ - \sum_{i=0}^{n-2} \gamma_i C_{n-i}(s+i) \\ \left. \right]$$

$$\text{where} \quad C_n(s) = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \left\{ \begin{array}{l} \text{all } k\text{-tuples from } \omega_2, \dots, \omega_{n-2} \text{ such} \\ \text{that no two adjacent } \omega_1 \text{'s are chosen,} \\ \text{form products and add.} \end{array} \right\}$$

This result is again proved by induction. The inductive argument is omitted as this approach is evidently hopeless and we did not pursue it further.

The reader may find it interesting that the corresponding equations for Erlang-r dead time distributions are still harder to handle than the ones for max-m impulse time.

Example 3:
$$H(x) = 1 - \sum_{j=0}^{r-1} \frac{e^{-\mu x} (\mu x)^j}{j!}$$

the basic equation (2.9)

$$A(y) = \int_0^y A(y-x)H(y-x)dF(x) + C(y) + \int_0^y \int_{y-x}^{\infty} A(z)dH(z)dF(x)$$

now becomes

$$(3.13) \quad A(y) = \int_0^y A(y-x) \left[1 - \sum_{j=0}^{r-1} \frac{e^{-\mu(y-x)} (\mu(y-x))^j}{j!} \right] dF(x) + C(y) \\ + \int_0^y \int_{y-x}^{\infty} A(z) \mu^r \frac{z^{r-1} e^{-\mu z}}{(r-1)!} dz dF(x)$$

Taking Laplace transforms, recalling that

$$\mathcal{L}(s) = \int_0^{\infty} e^{-sx} dF(x) \quad \mathcal{Y}(s) = \int_0^{\infty} e^{-sy} dA(y)$$

$$\text{and that} \quad \mathcal{M}(s) = \int_0^{\infty} e^{-sy} dC(y)$$

we obtain

$$\frac{\psi(s)}{s} = \frac{\Omega(s)}{s} \left[\frac{\psi(s)}{s} - \sum_{j=0}^{r-1} \frac{(-\mu)^j}{j!} \frac{d^j}{ds^j} \left[\frac{\psi(s+\mu)}{s+\mu} \right] + r(s) \right. \\ \left. + \frac{\Omega(s)}{s} \left\{ \frac{\mu^r (-1)^{r-1}}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} \left(\frac{\psi(s)}{s} \right) \Big|_{s=\mu} + \frac{(-\mu)^r}{r!} \frac{d^{r-1}}{ds^{r-1}} \left[\frac{\psi(s+\mu)}{s+\mu} \right] \right\} \right]$$

Reference

- [1] Takacs, L. (1957) On some probability problems in the theory of counters. Acta Math. Acad. Sci. Hung. 18, 127-138

§ 4.

Some Final Comments.

From the practical point of view we have not succeeded in obtaining useful formulas, however, we have given a situation where Erlang distributions are definitely not to be used. A further interesting observation is that the max-m distribution is the sum of m exponential distributions with parameters $m\mu, (m-1)\mu, \dots, \mu$. Thus a distribution which is the sum of non-identically distributed random variables was easier to handle than the Erlang-distribution, the sum of identically distributed random variables even in the integral equation approach. The reason for the success of the Erlang-distribution in the theory of the single server queue is that we can keep track of the "phase" of service (or arrival) because the customers are served in order. Thus the behavior of $E_k/G/1$ can be deduced from that of the single server queue with Poisson input and general service time where the service is in batches of size k . In the infinitely many server queue the order of the arrivals is of no help to us as service begins on arrival and the second arrival may well finish his service while the first is still in the queue. From the stand point of trying to reduce the general problem to one of bulk arrivals the max-m distributions would seem to be the distributions to use. As we have seen, unfortunately, they do not simplify the problem enough. We hope to have some more to say about these problems at some future date.

Chapter 6.

A Loss Function for an elementary missile defense system.

In a talk presented at Columbia in May 1961. Professor M.J. Hirsch discussed the problem of defending an island against an air attack with a one missile battery. He noted that the system has an "absorbing state" in the sense that if an attacking plane is not destroyed by the missile battery after time t_0 , it will be directly over the base and will attempt to destroy the base.

In this note we shall discuss the problem of determining the length of time until absorption (the base is destroyed) occurs assuming that:

- 1) the planes arrive according to a recurrent process, i.e., the inter-arrival times $\tau_{n+1} - \tau_n$ are independent, identically distributed positive random variables. $P[\tau_{n+1} - \tau_n \leq x] = F(x) \quad n=0,1,2,\dots$
- 2) The time it takes the missile battery to shoot a plane down is a positive random variable $H(x)$. If χ_n denotes the time it takes to shoot down the n-th plane then $P[\chi_n \leq x] = H(x)$ and $\{\chi_n\}_{n=1}^{\infty}$ are mutually independent and independent of the $\{\tau_n\}$ sequence. We differ from the usual queueing model in that we now assume that $P[\chi \geq t_0] = 1 - H(t_0) = P > 0$ where t_0 is the time we have to shoot the plane before it is in position to destroy the base.
- 3) We assume that we have infinitely many guns at the base but we use only one gun on each plane. However, we always have a gun

This approach is just that of taking a random number of random variables. The number N is given by a geometric distribution. This N denotes how many random variables $\theta_n = \tau_n - \tau_{n-1}$ are chosen. (see Feller [1]).

§2.

A slightly generalized Model

In the previous section we assumed that once the service time (total time used to shoot the plane) reached t_0 the plane destroyed the base. This corresponds in reality to perfect accuracy of the bombardier. Therefore we now assume that when the plane is over the base a bomb is dropped and with probability $C > 0$ it destroys the base. In order to determine $G(x)$ in this case we simply redefine p to equal $[1 - H(t_0)]C$ which now gives the probability that the n -th plane destroys the system.

Theorem 2:
$$\int_0^{\infty} e^{-sx} dG(x) = \frac{C[1 - H(t_0)] e^{-st_0}}{1 - C\mathcal{L}(s)[1 - H(t_0)]}$$

§3.

A decision theoretic criteria for efficiency.

In statistics one criteria for deciding upon a good estimate is to choose a loss function (usually squared error $L(\delta, \theta) = (\delta(x) - \theta)^2$) and using that estimate minimizing the expected loss. Unfortunately in most queueing problems it is very difficult to calculate these expected losses. However, for our model it is easy. A fairly realistic loss function would be of the form $L = \text{Loss} = kN + p(t)$ where k would be a constant such as cost per plane and $p(t)$ would be a polynomial in t (time).

But then

$$\begin{aligned} E(L) &= kE(N) + E(p(t)) \\ &= \frac{k}{p} + E(p(t)) \end{aligned}$$

But from the fact that $\int_0^{\infty} e^{-sx} dG(x) = \frac{pe^{-st_0}}{1-q(s)}$

we can calculate all the moments of the random variable t thus can calculate $E(p(t))$. For instance if $p(t) = t$

$$E(t) \text{ is } \frac{t_0 p + q\alpha}{p^2} \text{ and thus if } L = kN + t$$

$$E(L) = \frac{kp + t_0 p + q\alpha}{p^2} .$$

It is easily seen that the k -th moments of $G(x)$ depend only on p , t_0 and the first k moments of $F(x)$. Thus if all moments of $F(x)$ exist so do all the moments of $G(x)$.

References

- [1] Feller, William (1957) An Introduction to Probability Theory and its Applications 1 (2nd ed.) Wiley, New York
- [2] Hirsch, W.M., Conn, J. and Siegel, C. (1961) A queuing process with an absorbing state, Comm. on Pure and Applied Math. 14 137-153 .